Binomial Theorem Contractor **Free Constitution Binomial Theorem**

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C1 Binomial Expression :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example :
$$
x + y, x^2y + \frac{1}{xy^2}, 3 - x, \sqrt{x^2 + 1} + \frac{1}{(x^3 + 1)^{1/3}}
$$
 etc.

C2 Statement of Binomial theorem :

If x, $y \in R$ and $n \in N$, then : $(x + y)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1}b^1 + {}^nC_2 a^{n-2}b^2 + \dots + {}^nC_r a^{n-r}b^r + \dots + {}^nC_n a^0 b^n$

or $(x + y)^n = \sum_{r=0}^{n} C_r a^{n-r}$ **n r 0** $(\mathbf{x} + \mathbf{y})^{\mathbf{n}} = \sum_{\mathbf{r}}^{\mathbf{n}} \mathbf{C}_{\mathbf{r}} \mathbf{a}^{\mathbf{n} - \mathbf{r}} \mathbf{b}^{\mathbf{r}}$

Now, putting $y = 1$ in the binomial theorem

or
$$
(1+x)^n = {}^nC_0 + {}^nC_1 + {}^nC_2x^2 + \dots + {}^nC_rx^r + \dots + {}^nC_nx^n
$$

$$
(1+x)^n = \sum_{r=0}^n {^nC_r}x^r
$$

Practice Problems :

- **1. Using binomial theorem, indicate which number is larger** $(1.1)^{10000}$ **or 1000.**
- **2. Find** $(x + 1)^6 + (x 1)^6$. Hence or otherwise evaluate $(2 + 1)^6 + (2 1)^6$.
- 3. **Show that** $9^{n+1} 8n 9$ **is divisible by 64, whenever n is a positive integer.**
- **Figure 1) From Solution From Solution Free Chaota Control Free Chaota Control Free Controllering 1 Controllering 1 Whereform is n.**
 Free Solution Free Controllering 1 Controllering 1 Whereform is n. 4. Using binomial theorem, prove that 6ⁿ – 5n always leaves remaining 1 when divided by 25.

C3 Properties of Binomial Theorem :

- (i) The number of terms in the expansion is $n + 1$.
- **(ii)** The sum of the indices of x and y in each term is n.
- (iii) The binomial coefficients (nC_0 , nC_1 nC_n) of the terms equidistant from the
- begining and the end are equal, i.e. ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$ etc.
- **C4** Some important terms in the expansion of $(x + y)^n$:

(i) General term :

- $(r + 1)$ th term of $(x + y)^n$ is $T_{r+1} = {}^nC_r x^{n-r} y^r$
- **(ii) Middle term/(s) :**

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or
$$
(x + y)^n = \sum_{r=0}^{n} C_r a^{n-r} b^r
$$

\nNow, putting $y = 1$ in the binomial theorem
\nor $(1 + x)^n = C_0 + C_1 + C_2x^2 + + C_cx^r + + C_cx^n$
\n $(1 + x)^n = \sum_{r=0}^{n} {^n}C_r x^r$
\n**Practice Problems :**
\nUsing binomial theorem, indicate which number is larger $(1.1)^{\text{(60)}}$ or 1000.
\nFind $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $(2 + 1)^6 + (2 - 1)^6$.
\nShow that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.
\nUsing binomial theorem, prove that $6^n - 5n$ always leaves remaining 1 when divided by 25.
\n**Properties of Binomial Theorem :**
\n(i) The number of terms in the expansion is $n + 1$.
\n(ii) The form of the indices of x and y in each term is n.
\n(iii) The binomial coefficient $(C_0, C_1,, C_0) \neq 0$ the terms equidistant from the
\nbegining and the end are equal, i.e. $C_0 = C_2,, C_0$ if the terms equidistant from the
\nbegining and the end are equal, i.e. $C_0 = C_2,, C_0$ if the terms
\n(i) General term:
\n(ii) Middle term/(s):
\n(i) Find the term/(s) :
\n(ii) Middle term/(s):
\n(i) if n is odd, there are two middle terms, which are
\n $\left(\frac{n+1}{2}\right)$ th and $\left(\frac{n+1}{2}+1\right)$ th terms.

if n is odd, there are two middle terms, which are

$$
\left(\frac{n+1}{2}\right)
$$
th and $\left(\frac{n+1}{2}+1\right)$ th terms.

(iii) **Numerically greatest term in the expansion of** $(x + y)^n$ **,** $n \in N$

Let T_r and T_{r+1} be the rth and $(r + 1)$ th terms respectively

$$
\begin{array}{ccc} T_r & = & {}^{n}C_{r-1} \; x^{n-(r-1)} \; y^{r-1} \\ T_{r+1} & = & {}^{n}C_{r} \; x^{n-r} \; y^{r} \end{array}
$$

Now,

Consider **1**

$$
\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^{n}C_r}{{}_{n}C_{r-1}} \frac{{}^{x^{n-r}y^{r}}}{x^{n-r+1}y^{r-1}}\right| = \frac{n-r+1}{r} \cdot \left|\frac{y}{x}\right|
$$

$$
\left|\frac{T_{r+1}}{T_r}\right| \ge 1, \left(\frac{n-r+1}{r}\right) \left|\frac{y}{x}\right| \ge 1, \frac{n+1}{r} - 1 \ge \left|\frac{x}{y}\right|, \ r \le \frac{n+1}{1 + \left|\frac{x}{y}\right|}
$$

n r r

-

Practice Problems :

1. Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{2x^2+2x}{n!}$ $(1+x)^{2n}$ is $\frac{1\cdot3\cdot5...(2n-1)}{n}$ \cdot $2^n\cdot x^n$, where $n \in \mathbb{N}$.

r n

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- **2. Show that the coefficient of the middle term in the expansion of** $(1 + x)^{2n}$ **is the sum of the coefficients of two middle terms in the expansion of** $(1 + x)^{2n-1}$ **.**
- **3. Find the value of r, if the coefficients of** $(2r + 4)$ **th and** $(r 2)$ **th terms in the expansion of** $(1 + x)^{18}$ **are equal.**
- **4. If the coefficient of** $(r 1)$ **th, rth and** $(r + 1)$ **th terms in the expansion of** $(x + 1)$ **ⁿ are in the ratio 1 : 3 : 5, find n and r**
- **5. The 2nd, 3rd and 4th terms in the expansion of** $(x + y)^n$ **are 240, 720 and 1080 respectively. Find the values of x, y and n.**
- **6. Find the coefficient of** x^5 **in the product** $(1 + 2x)^6 (1 x)^7$ **using binomial theorem.**
- **7. Find the term independent of x in the expansion of 6 2 3x** $\frac{3}{2}x^2 - \frac{1}{3}$ $\frac{3}{2}x^2-\frac{1}{2}$ Į $\left(\frac{3}{2}x^2-\frac{1}{2}\right)$ Ņ $\left(\frac{3}{2}x^2-\frac{1}{2}\right)^{\circ}$.
- **8. Find the coefficient of a⁴ in the product** $(1 + 2a)^4 (2 a)^5$ **using binomial theorem.**
- 2. Show that the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is the sum of the solution of the expansion of $(1+x)^{2n+1}$.

3. Find the value of r, if the coefficients of $(2r + 4)$ th and $(r 2)$ th terms i of x in the expansion of $\left(\frac{-x}{2} - \frac{y}{3x}\right)$.

the product $(1 + 2a)^4 (2 - a)^5$ using binomial the product $(1 + 2a)^4 (2 - a)^5$ using binomial the ferm of the expansion containing x³.

denotes the expansion of $\left(x + \frac{1}{x}\right)^$ **9. The sum of the coefficients of the first three terms in the expansion of** $\left| \mathbf{x} - \frac{\mathbf{b}}{2} \right|$ **,** $\mathbf{x} \neq 0$ **x** $\mathbf{x} - \frac{3}{4}$ **m** $\frac{1}{2}$, x \neq J $\left(x-\frac{3}{2}\right)$ J $\left(x-\frac{3}{2}\right)^{m}$, $x \neq 0$, m being a **natural number, is 559. Find the term of the expansion containing x³ .**

10. Show that the greatest coefficients in the expansion of
$$
\left(x + \frac{1}{x}\right)^{2n}
$$
 is $\frac{1 \cdot 3 \cdot 5 \cdot ... \cdot (2n-1) \cdot 2^n}{n!}$.

- **11. Express** $(x + \sqrt{x^2 + 1})^6 + (x \sqrt{x^2 + 1})^6$ as a polynomial in x.
- **12.** If a_1 , a_2 , a_3 and a_4 be any four consecutive coefficients in the expansion of $(1 + x)^n$, prove that

$$
\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{2a_2}{a_2 + a_3}.
$$

[Answers : (3) 6 (4) n = 7, r = 3 (5) x = 2, y = 3 and n = 5 (6) 171 (7) 5/12 (8) –438 (9) –5940 x³]

C5 Multinominal Theorem

As we know the Binomial Theorem $(x + y)^n = \sum_{r=0}^{n} {^{n}C_r x^{n}}$ **n r 0** $(\mathbf{x} + \mathbf{y})^{\mathbf{n}} = \sum_{\mathbf{r}=0}^{\mathbf{n}} \mathbf{C}_{\mathbf{r}} \mathbf{x}^{\mathbf{n}-\mathbf{r}} \mathbf{y}^{\mathbf{r}} = \sum_{\mathbf{r}=0}^{\mathbf{n}} \frac{\mathbf{n} \cdot \mathbf{r}}{(\mathbf{n}-\mathbf{r})!\mathbf{r}!} \mathbf{x}^{\mathbf{n}-\mathbf{r}}$ ÷, = **n r 0** $\mathbf{x}^{\mathbf{n-r}}\mathbf{y}^{\mathbf{r}}$ $(n - r)!r!$ **n!**

putting $n - r = r_1$, $r = r_2$

therefore, $(\mathbf{x} + \mathbf{y})^n = \sum_{\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}_1}$ $\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{n}$ \mathbf{r}_1 **r r 1 2 n 1 2** $\mathbf{x}^{\mathbf{r}_1}$ $\mathbf{y}^{\mathbf{r}_2}$ ${\bf r}_1!{\bf r}_2!$ $(x+y)^n = \sum_{ } \frac{\mathbf{n}!}{ }$

Total number of terms in the expansion of $(x + y)^n$ is equal to number of non-negative integral solution of r_1 + r₂ = n i.e. $n+2-1C_{2-1} = n+1C_1 = n+1$

In the same fashion we can write the multinominal theorem

$$
(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \dots \mathbf{x}_k)^n = \sum_{r_1 + r_2 + \dots + r_k = n} \frac{n!}{r_1! r_2! \dots r_k!} \mathbf{x}_1^{r_1} \cdot \mathbf{x}_2^{r_2} \dots \mathbf{x}_k^{r_k}
$$

Here total number of terms in the expansion of $(x_1 + x_2 + + x_k)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 + \dots + r_k = n$ i.e. $n+k-1C_{k-1}$

Practice Problems :

1. (i) the middle term in the expansion of 10 2y $\mathbf{x} - \frac{1}{2\mathbf{v}}$ Į ſ l I $\left(x - \frac{1}{2}\right)^{10}$ (ii) the coefficient of x^{32} and x^{-17} in the expansion

of
$$
\left(x^4 - \frac{1}{x^3}\right)^{15}
$$

2. Find the coefficient of x^5 **in the expansion of the product** $(1 + 2x)^5 (1 - x)^7$ **.**

[Answers : (1) (i)
$$
-\frac{63x^5}{8y^5}
$$
 (ii) 1365, -1365 (2) 171]

C6 Properties of Binomial Coefficients :

of $\left(x^4 - \frac{1}{x^3}\right)^{1/5}$
 Find the coefficient of x' in the expansion of the product $(1 + 2x)^3 (1 - x)^7$.
 (Answers : (1) (i) $-\frac{63x^5}{8y^5}$ (ii) 1365, -1365 (2) 171]
 EVALUATE:
 We are the product of the expansi FRACE ACT SET ASSESS
 **FRACE 2ⁿ

FC**_n = 2ⁿ
 FC_n = 2ⁿ
 FC_n = 2ⁿ
 FC_n = 0
 FC_n $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_rx^r + \dots + C_nx^n$(1) (1) The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n Putting $x = 1$ in (1) ${}^{n}C_{0} + {}^{n}C_{1} + {}^{n}C_{2} + \dots + {}^{n}C_{n} = 2^{n}$(2) or $\sum_{r=0}^{n} C_r =$ **n r 0 n** $^{\mathrm{n}}$ **C**_r = 2 (2) Again putting $x = -1$ in (1), we get ${}^{n}C_{0} - {}^{n}C_{1} + {}^{n}C_{2} - {}^{n}C_{3} + \dots + (-1)^{n} {}^{n}C_{n}$ $....(3)$ or $\sum_{r=0}^{\infty} (-1)^{r} C_r =$ **n r 0 r** $(-1)^{r} C_r = 0$ (3) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1} i.e., ${}^{n}C_{0} + {}^{n}C_{2} + {}^{n}C_{4} + \dots \dots \dots \dots \dots = 2^{n-1}$ ⁿC¹ + ⁿC³ + ⁿC⁵ + = 2n – 1 Sum of two consecutive binomial coefficients ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

(5) Ratio of two consecutive binomial coefficients $\frac{d\mathbf{r}}{d\mathbf{C}_{r-1}} = \frac{d\mathbf{r}}{d\mathbf{r}}$ $n - r + 1$ **C C r 1 n r** $\frac{{}^{n}C_{r}}{C_{n}}=\frac{n-r+1}{r}$ -

(6)
$$
{}^{n}C_{r} = \frac{n}{r} {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)} {}^{n-2}C_{r-2}
$$

Practice Problems :

1. Prove the following identities :

- (a) ${}^{n}C_0 + {}^{n}C_2 + {}^{n}C_4 + \dots = 2^{n-1}$
- **(b)** ${}^{n}C_{1} + {}^{n}C_{3} + {}^{n}C_{5} + \dots = 2^{n-1}$
- (c) ${}^{n}C_{0} + 3 {}^{n}C_{1} + 5 {}^{n}C_{2} + \dots + (2n + 1) {}^{n}C_{n} = (n + 1)2^{n}$
- (d) ${}^{n}C_{1} 2 {}^{n}C_{2} + 3 {}^{n}C_{3} \dots + (-1)^{n-1} {}^{n}C_{n} = 0$
- (e) $C_1 + 2C_2 + 3C_3 + \dots + nC_n = n2^{n-1}$
- **(f)** $C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n = 2^n + n2^{n-1}$
- **(g) n 1** $\frac{C_2}{3}$ – = $\frac{1}{n+1}$ **C 2** $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots = \frac{1}{n+1}$

(h)
$$
2C_0 + 2^2 \frac{C_1}{2} + 2^3 \frac{C_2}{3} + \dots + 2^{n+1} \frac{C_n}{n+1} = \frac{3^{n+1} - 1}{n+1}
$$

(i)
$$
C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \dots = \frac{2^n}{n+1}
$$

r!

(j)
$$
2C_0 + 5 C_1 + 8 C_2 + \dots + (3n + 2) C_n = (3n + 4) 2^{n-1}
$$

C7 Binomial Theorem For Negative Integer or Fractional Indices If $n \in R$ then,

(h)
$$
2C_0 + 2^2 \frac{C_1}{2} + 2^3 \frac{C_2}{3} + \dots + 2^{n+1} \frac{C_n}{n+1} = \frac{3^{n+1}-1}{n+1}
$$

\n(i) $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \dots = \frac{2^n}{n+1}$
\n(j) $2C_0 + 5 C_1 + 8 C_2 + \dots + (3n+2) C_n = (3n+4) 2^{n-1}$
\n**Binomial Theorem For Negative Integer or Fractional Indices**
\nIf $n \in \mathbb{R}$ then,
\n $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$
\n $+ \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots$
\n**Remarks**
\n(i) The above expansion is valid for any rational number other then a whole number if $|x| < 1$.
\n(iii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1 + x)^n$ is infinite, and the symbol nC_r cannot be used to denote the coefficient of the general term.
\n(ii) The first terms must be unity in the expansion, when index 'n' is a negative integer or fraction.
\n(iv) The general term in the expansion of $(1+x)^n$ is $T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$
\n(v) When 'n' is any rational number other than whole number then approximate value of $(1+x)^n$ is $1 + nx(x^2)$ and higher powers of x can be neglected)
\n(v) Expansion to be remembered $(|x| < 1)$
\n(a) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots + (-1)^r x^r + \dots + (-1)^r (r+1) x^r + \dots + (-1)^r x^r + \dots$

Remarks

- (i) The above expansion is valid for any rational number other then a whole number if $|x| < 1$.
- (ii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1 + x)^n$ is infinite. and the symbol nC_r cannot be used to denote the coefficient of the general term.
- (iii) The first terms must be unity in the expansion, when index 'n' is a negative integer or fraction.

(iv) The general term in the expansion of
$$
(1+x)^n
$$
 is $T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$

(v) When 'n' is any rational number other than whole number then approximate value of $(1 + x)^n$ is $1 + nx$ (x^2 and higher powers of x can be neglected)

(vi) Expansion to be remembered $(|x| < 1)$

- (a) $(1 + x)^{-1} = 1 x + x^2 x^3 + \dots + (-1)^{r} x^{r} + \dots + \infty$
- (b) $(1-x)^{-1} = 1 + x + x^2 + x^5 + \dots + x^r + \dots \dots \dots \infty$
- (c) $(1 + x)^{-2} = 1 2x + 3x^2 4x^3 + \dots + (-1)^{r} (r + 1) x^{r} + \dots +$
- (d) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots + (r+1)x^r + \dots \dots \infty$

Practice Problems :

- **1. Find the coefficient of** x^6 **in the expansion of** $(1 2x)^{-5/2}$ **.**
- **2. Find the coefficient of** x^{10} **in the expansion of** $\frac{(x+bx)^2}{(1-x^2)^3}$ **2** $(1 - x^2)$ $(1+3x^2)$ - $+\frac{3x^2}{x^3}$, mentioning the condition under which the

result holds.

[Answers : (1)
$$
\left[\frac{15015}{16}\right]
$$
]