Complex Number

C1 The complex number

Complex number is denoted by z i.e. z = a + ib, where 'a' is called as real part of z (denoted by Re z) and 'b' is called as imaginary part of z (denoted by Im z). Here $i = \sqrt{-1}$, also $i^2 = -1$, $i^3 = -i$; $i^4 = 1$ etc.

The set R of real numbers is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

Practice Problems :

1. If n is natural number then the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is

(a) 1 **(b)** 0 **(c)** i **(d)** –i

2. The value of $(i^{100} + 1)(i^{99} + 1)....(i + 1)$ will be

(a) 0 **(b)** 1 **(c)** i **(d)** –i

[Answers : (1) b (2) a]

C2 Algebraic Operations on Complex Number :

1. Addition $(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$
2. Subtraction $(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$
3. Multiplication $(a + ib)(c + id) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$
4. Division $\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$
5. Inequalities in complex numbers are not defined.
6. In real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$ however in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.
7. Equality In Complex Number : If $z_1 = z_2 \Rightarrow Re(z_1) = Re(z_2)$ and $I_m(z_1) = I_m(z_2)$

Practice Problems :

1. The value of $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$ is equal to

(a) $\frac{1}{2} + \frac{9}{2}i$ **(b)** $\frac{1}{2} - \frac{9}{2}i$ **(c)** $\frac{1}{4} - \frac{9}{4}i$ **(d)** $\frac{1}{4} + \frac{9}{4}i$

2. If $\left(\frac{1+i}{1-i}\right)^m = 1$ then the least integral value of m is

(a) 2 **(b)** 4 **(c)** 8 **(d)** 10

3. If $\frac{(1+i)x - 2i}{3+i} + \frac{(2+3i)y + i}{3-i} = i$, then the real value of x and y are given by

(a) x = –3, y = –1 **(b)** x = 3, y = –1 **(c)** x = 3, y = 1 **(d)** x = 1, y = –3

[Answers : (1) d (2) b (3) b]

C3 Modulus of a Complex Number :

If z = a + ib, then it's modulus is denoted and defined by $|z| = \sqrt{a^2 + b^2}$. Infact |z| is the distance of z from origin.

Properties of modulus

(i) $|z_1 z_2| = |z_1||z_2|$ (ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ (provide $z_2 \neq 0$)

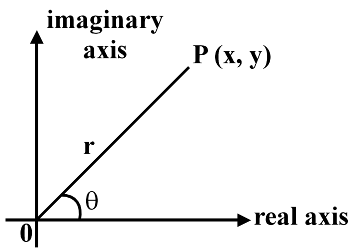
(iii) $|z_1 + z_2| \leq |z_1| + |z_2|$ (iv) $|z_1 - z_2| \geq ||z_1| - |z_2||$

(Equality in (iii) and (iv) holds if and only if origin, z_1 and z_2 are collinear with z_1 and z_2 on the same side of origin).

C4 Representation of a Complex Number :

(a) Cartesian Form (Geometric Representation) :

Every complex number z = x + i y can be represented by a point on the Cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y)



Length OP = $|z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

θ is called the argument or amplitude. If θ is the argument of a complex number then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. The unique value of θ such that $-\pi < \theta \leq \pi$ is called the principal value of the argument. Unless otherwise stated, amp z implies principal value of the argument.

The argument of z = θ, π – θ, – π + θ, – θ, $\theta = \tan^{-1}\left|\frac{y}{x}\right|$, according as z = x + iy lies in I, II, III or IVth quadrant.

Properties of Argument of a Complex Number :

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2m\pi$ for some integer m.

(ii) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2) + 2m\pi$ for some integer m.

(iii) $\arg(z^2) = 2\arg(z) + 2m\pi$ for some integer m.

(iv) $\arg(z) = 0$ ⇔ z is real, for any complex number $z \neq 0$

(v) $\arg(z) = \pm \pi/2$ ⇔ z is purely imaginary, for any complex number $z \neq 0$

(vi) $\arg(z_2 - z_1)$ = angle of the line segment joining the point (z_1) and point (z_2)

(b) Trignometric/Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

$\cos\theta + i\sin\theta$ is also written as CiS θ or $e^{i\theta}$.

(c) Euler's Representation :

$$\mathbf{z = re^{i\theta}; |z| = r; \arg z = \theta; \bar{z} = re^{-i\theta}}$$

Also $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2}$ are known as Euler's identities.

(d) Vectorial Representation :

Every complex number can be considered as if it is the position vector of a point. If the point P represents the complex number z then, $\overrightarrow{OP} = z \;\&\; |\overrightarrow{OP}| = |z|$.

Practice Problems :

1. If $|z| = 4$ and $\arg z = \frac{5\pi}{6}$ then z equals to

(a) $2\sqrt{3} - 2i$ (b) $2\sqrt{3} + 2i$ (c) $-2\sqrt{3} + 2i$ (d) $-\sqrt{3} + i$

2. If $x_r = \cos\left(\frac{\pi}{2^r}\right) + i\sin\left(\frac{\pi}{2^r}\right)$ then

$x_1 \cdot x_2 \cdot x_3 .... \infty$ is

(a) –3 (b) –2 (c) –1 (d) 0

3. The amplitude or argument of $\frac{(1+i)(2+i)}{(3-i)}$ will be

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

[Answers : (1) c (2) c (3) c]

C5 Conjugate of a complex Number

Conjugate of a complex number z = a + ib is denoted and defined by $\bar{z} = a - ib$.

Properties of conjugate

(i) $|z| = |\bar{z}|$ (ii) $z\bar{z} = |z|^2$ (iii) $\overline{(z_1 + z_2)} = (\bar{z}_1) + (\bar{z}_2)$

(iv) $\overline{(z_1 - z_2)} = (\bar{z}_1) - (\bar{z}_2)$ (v) $\overline{(z_1 z_2)} = (\bar{z}_1)\,(\bar{z}_2)$ (vi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(\bar{z}_1)}{(\bar{z}_2)} (z_2 \neq 0)$

(vii) $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$ (viii) $\overline{(\bar{z})} = z$

(ix) $\arg(z) + \arg(\bar{z}) = 0$

C6 Demoivere's Theorem :

If n is any integer then

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)....$
$(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + ... + \theta_n) + i\sin(\theta_1 + \theta_2 + ... + \theta_n)$

Practice Problems :

1. Simplify the following :

(i) $\frac{(\cos 2\theta - i\sin 2\theta)^7 (\cos 3\theta + i\sin 3\theta)^5}{(\cos 5\theta + i\sin 5\theta)^3 (\cos 7\theta - i\sin 7\theta)^2}$ (ii) $\frac{i^5(\sin 2\theta + i\cos 2\theta)}{(\cos\theta - i\sin\theta)^2}$

2. If $x = \cos\theta + i\sin\theta$ and $\sqrt{1-c^2} = nc - 1$, show that $1 + c\cos\theta = \frac{c}{2n}(1+nx)\left(1+\frac{n}{x}\right)$.

3. If $x + (1/x) = 2\cos\theta$ and $y + (1/y) = 2\cos\phi$ etc, then prove that

(i) $xyz.... + \frac{1}{xyz}.... = 2\cos(\theta+\phi+...)$ **(ii)** $\frac{x}{y} + \frac{y}{x} = 2\cos(\theta-\phi)$

(iii) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$ **(iv)** $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$

[Answers : 1 (i) 1 (ii) –1]

C7 Cube Root of Unity :

(i) The cube root of unity are $\mathbf{1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}}$

(ii) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

(iii) In polar form the cube roots of unity are : $\mathbf{\cos 0 + i\sin 0; \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}}$

(iv) The three cube roots of unity when plotted on the argand plane constitute the varties of an equilateral triangle.

(v) The following factorisation should be remembered :

(a, b, c $\in$ R and ω is the cube root of unity)

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $a^2 + a + 1 = (a - \omega)(a - \omega^2)$

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$; $a^2 + ab + b^2 = (a - b\omega)(a - b\omega^2)$

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

Practice Problems :

1. If 1, ω, ω^2 are the cube roots of unity, prove that

(i) $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$ **(ii)** $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) = 1$

(iii) $(2 + 5\omega + 2\omega^2)^6 = (2 + 2\omega + 5\omega^2)^6 = 729$

(iv) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$.........**to 2n factors** $= 2^{2n}$.

(v) $1 + \omega^n + \omega^{2n} = \begin{cases} 3, \text{when n is a multiple of 3} \\ 0, \text{when n is not a multiple of 3} \end{cases}$

(vi) $(a + \omega + \omega^2)(a + \omega^2 + \omega^4)(a + \omega^4 + \omega^8)$....**to 2n factors.** $= (a - 1)^{2n}$

(vii) $\frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2} + \frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} = -1$

2. If ω and ω^2 are complex cube roots of unity, prove that

(i) $x^3 + y^3 = (x + y)(\omega x + \omega^2 y)(\omega^2 x + \omega y)$ **(ii)** $x^3 - y^3 = (x - y)(\omega x - \omega^2 y)(\omega^2 x - \omega y)$

3. If ω is an imaginary cube root of unity, prove that $\frac{1}{1+2\omega} + \frac{1}{2+\omega} - \frac{1}{1+\omega} = 0$.

4. Given $z_1 + z_2 + z_3 = A$, $z_1 + z_2\omega + z_3\omega^2 = B$, $z_1 + z_2\omega^2 + z_3\omega = C$ where ω is cube root of unity

(i) express z_1, z_2, z_3 in terms of A, B, C

(ii) prove that $|A|^2 + |B|^2 + |C|^2 = 3(|z_1|^2 + |z_2|^2 + |z_3|^2)$

[Answers : (4) $z_1 = \frac{A+B+C}{3}$, $z_2 = \frac{A+B\omega^2+C\omega}{3}$, $z_3 = \frac{A+B\omega+C\omega^2}{3}$ **]**

C8 nth Roots of Unity :

If 1, $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots\alpha_{n-1}$ are the n, nth root of unity then :

(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$

(ii) $1^p + \alpha_1^p + \alpha_2^p + \ldots + \alpha_{n-1}^p = 0$, if p is not an integral multiple of n

$= n$ if p is an integral multiple of n

(iii) $(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_{n-1}) = n$

$(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

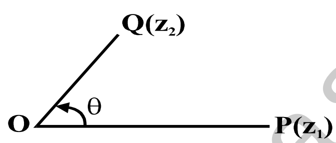
(iv) $1 . \alpha_1 . \alpha_2 . \alpha_3 \ldots\ldots\alpha_{n-1} = 1$ or -1 according as n is odd or even.

Practice Problems :

1. **If 1, $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the nth roots of unity and n is an odd natural number then find the value of $(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)\ldots(1+\alpha_{n-1})$.**

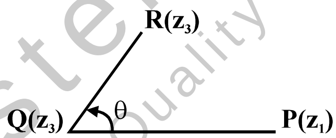
C9 Rotation theorem

(i) If $P(z_1)$ and $Q(z_2)$ are two complex numbers such that $|z_1| = |z_2|$, then $z_2 = z_1e^{i\theta}$ where $\theta = \angle POQ$



(ii) If $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then

$$\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \left|\frac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$$



Practice Problems :

1. **If z_1, z_2, z_3 are vertices of an equilateral Δ having its circumcentre at origin such that $z_1 = 1 + i$ then find z_2 and z_3.**

2. **Show that the triangle whose vertices are the points represented by the complex numbers z_1, z_2, z_3 on the Argand plane is equilateral if and only if $\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$, that is if and only if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.**

3. **If z_1, z_2, z_3 be the affixes of the vertices A, B and C respectively of a triangle ABC having centroid at G. such that z = 0 is the mid-point of AG, then prove that $4z_1 + z_2 + z_3 = 0$.**

4. **(a) Complex numbers z_1, z_2, z_3 are the vertices A,B,C respectively of an isosceles right angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$**

   (b) If $z_1^2 + z_2^2 - 2z_1z_2 \cos\theta = 0$ then the origin z_1, z_2 form vertices of an isosceles triangle with vertical angle θ.

5. Show that the triangle whose vertices are z_1, z_2, z_3 and z_1', z_2', z_3' are directly similar if

$$\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0 .$$

C10 Logarithm of a Complex Quantity :

(i) $$\text{Log}_e(\alpha + i\beta) = \frac{1}{2}\text{Log}_e(\alpha^2 + \beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right) \text{ where } n \in I$$

(ii) i^i represents a set of positive real numbers given by $e^{-\left(2n\pi + \frac{\pi}{2}\right)}, n \in I$

C11 Geometrical Properties :

(i) **Distance Formula :** If z_1 and z_2 are affixies of the two points P and Q respectively then distance between P and Q is given by $|z_1 - z_2|$

(ii) **Section Formula :** If z_1 and z_2 are affixes of the two points P and Q respectively and point C divides the line joining P and Q internally in the ratio m : n then affix z to C is given by

$$z = \frac{mz_2 + nz_1}{m + n}$$

If C divides PQ in the ratio m : n externally then $z = \frac{mz_2 - nz_1}{m - n}$

(iii) **Condition of collinearty :**

If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where a + b + c = 0 and a, b, c are not all simultaneoulsy zero, then the complex numbers z_1, z_2 and z_3 are collinear.

Important Results :

(1) If the vertices A, B, C of a triangle represents the complex numbers z_1, z_2, z_3 respectively and a, b, c are the length of sides then,

(i) Centroid of the ΔABC = $\frac{z_1 + z_2 + z_3}{3}$

(ii) Orthocenter of the ΔABC =

$$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C} \text{ or } \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$

(iii) Incentre of the ΔABC = $(az_1 + bz_2 + cz_3) / (\sin 2A + \sin 2B + \sin 2C)$

(2) amp (z) = θ is a ray emanating from the origin inclined at an angle θ to the x-axis.

(3) $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line joining z_1 to z_2.

(4) The equation of a line joining z_1 and z_2 is given by, $z = z_1 + t(z_1 - z_2)$ where t is a real parameter.

(5) $z = z_1(1 + it)$ where t is a real parameter is a line through the point z_1 & perpendicular to the line joining z_1 to the origin.

(6) The equation of a line passing through z_1 and z_2 can be expressed in the determinant form as

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 .$$ This is also the condition for three complex numbers to be collinear. The

above equation on manipulating, takes the form $\bar{\alpha}z + \alpha\bar{z} + r = 0$ where r is real and α is a non zero complex constant.

(7) The equation of circle having centre z_0 and radius r is : $|z - z_0| = r$
General equation of the circle $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ where a is a complex number and b is real number. Centre of the circle is –a and radius is $\sqrt{|a|^2 - b}$.

(8) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is

$$\arg\frac{z - z_2}{z - z_1} = \pm\frac{\pi}{2} \text{ or } (z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0 .$$

(9) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is the number $\frac{z_3 - z_1}{z_3 - z_2}\cdot\frac{z_4 - z_2}{z_4 - z_1}$ should be real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as $\frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$ is real

$$\Rightarrow \quad \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \frac{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}$$

(10) $\operatorname{Arg}\left(\frac{z - z_1}{z - z_2}\right) = \theta$ represent

(i) a line segment if $\theta = \pi$

(ii) Pair of ray if $\theta = 0$

(iii) a part of circle, if $0 < \theta < \pi$

(11) Area of triangle formed by the points z_1, z_2 & z_3 is $\left|\frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|$

(12) Perpendicular distance of a point z_0 from the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is $\frac{|\bar{\alpha}z_0 + \alpha\bar{z}_0 + r|}{2|\alpha|}$

(13) (i) Complex slope of a line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is $\omega = -\frac{\alpha}{\bar{\alpha}}$

(ii) Complex slope of a line joining by the points z_1 & z_2 is $\omega = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

(iii) Complex slope of a line makine θ angle with real axis = $e^{2i\theta}$

(14) ω_1 & ω_2 are the complex slopes of two lines.

(i) If lines are parallel then $\omega_1 = \omega_2$

(ii) If lines are perpendicular then $\omega_1 + \omega_2 = 0$

(15) If $|z - z_1| + |z - z_2| = K > |z_1 - z_2|$ then locus of z is an ellipse whose focii are z_1 & z_2

(16) If $|z - z_0| = \left|\frac{\bar{\alpha}z + \alpha\bar{z} + r}{2|\alpha|}\right|$ then locus of z is parabola whose focus is z_0 and directrix is the line

$\bar{\alpha}z_0 + \alpha\bar{z}_0 + r = 0$

(17) If $\left|\frac{z-z_1}{z-z_2}\right| = k$ where $k \neq 0$ or 1 then locus of z is circle.

(18) If $||z-z_1| - |z-z_2|| = K < |z_1 - z_2|$ then locus of z is a hyperbola, whose focii are z_1 & z_2.

C12 (a) **Reflection points for a straight line :**

Two given points P & Q are the reflection points for a given straight line if the given line is the right bisector of the segment PQ. Note that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if;

$\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.

(b) **Inverse points w.r.t. a circle :**

Two points P & Q are said to be inverse w.r.t. a circle with centre 'O' and radius ρ, if :

(i) the point O, P, Q are collinear and P, Q are on the same side of O.

(ii) OP . OQ = ρ^2.

The two points z_1 & z_2 will be the inverse points w.r.t. the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if $z_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$.

<u>Practice Problems :</u>

1. **Find the radius and centre of the circle $z\bar{z} + (1-i)z + (1+i)\bar{z} - 7 = 0$.**
2. **Determine the value of k for which equation $z\bar{z} + (-3+4i)\bar{z} - (3+4i)z + k = 0$ represent a circle.**
3. **Show that the points representing the complex numbers (3 + 2i), (2 – i) and –7i are collinear.**
4. **Find the perpendicular bisector of 3 + 4i and –5 + 6i.**
5. **If z_1, z_2, z_3 are the affixes of the vertices of a triangle having its circumcentres at the origin. If z is the affix of its orthocentre then prove that $z_1 + z_2 + z_3 - z = 0$**
6. **Find the locus of a complex number z in the Argand plane, satisfying $|z - (1 + i)| = 5$.**
7. **Show that the locus of a complex number z satisfying $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$ is a circle. Find the equation of the circle in cartesian coordinates.**
8. **Locate the points in the Argand plane representing the complex numbers z = x + iy for which**

   (i) $|z + 1| + |z - 1| < 3$

   (ii) $\arg(z - 4 - i) = \frac{\pi}{6}$

   (iii) $|z - 1| + |z + 1| = 4$

   (iv) $\arg(z + i) - \arg(z - i) = \pi/2$
9. **Find the locus of the complex number z in the Argand plane if $\left|\frac{1-iz}{z-i}\right| = 1$.**
10. **If z = x + iy and $\omega = \frac{1-zi}{z-i}$, $|\omega| = 1$, then find the locus of z.**

[Answers : (1) (–1, –1), 3 (2) k ≤ 25 (4) $(8+2i)z + (8-2i)\bar{z} + 36 = 0$ (6) Circle (7) $x^2 + y^2 = 1$ (8) (i) Interior of the ellipse having foci at (1, 0) and (–1, 0) and major axis of length 3 units (ii) A straight line passing through (4, 1) and making an angle of $\pi/6$ with x-axis (iii) Ellipse with foci at 1 + 0.i and –1 + 0.i and centre at origin (iv) Locus of point z is a circle with diameter AB and centre at origin with radius 1 (10) z lies on the real axis]

C13 **Ptolemy's Theorem :**

It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the products of lengths of the two pairs of its opposite sides.

i.e. $|z_1 - z_3|\,|z_2 - z_4| = |z_1 - z_2|\,|z_3 - z_4| + |z_1 - z_4|\,|z_2 - z_3|$.