Binomial Theorem

BINOMIAL THEOREM

C1 Binomial Expression :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $\mathbf{x+y, x^2y+\frac{1}{xy^2}, 3-x, \sqrt{x^2+1}+\frac{1}{(x^3+1)^{1/3}}}$ **etc.**

C2 Statement of Binomial theorem :

If x, y ∈ R and n ∈ N, then :

$(x+y)^n = {}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1}b^1 + {}^nC_2 a^{n-2}b^2 + ... + {}^nC_r a^{n-r}b^r + ... + {}^nC_n\, a^0b^n$

or $$(x+y)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r$$

Now, putting y = 1 in the binomial theorem

or $(1+x)^n = {}^nC_0 + {}^nC_1 + {}^nC_2x^2 + ... + {}^nC_r x^r + ... + {}^nC_n x^n$

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$$

<u>Practice Problems :</u>

1. **Using binomial theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.**
2. **Find $(x+1)^6 + (x-1)^6$. Hence or otherwise evaluate $(2+1)^6 + (2-1)^6$.**
3. **Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.**
4. **Using binomial theorem, prove that $6^n - 5n$ always leaves remaining 1 when divided by 25.**

C3 Properties of Binomial Theorem :

(i) The number of terms in the expansion is n + 1.

(ii) The sum of the indices of x and y in each term is n.

(iii) The binomial coefficients (${}^nC_0, {}^nC_1 ....{}^nC_n$) of the terms equidistant from the begining and the end are equal, i.e. ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$ etc.

C4 Some important terms in the expansion of $(x+y)^n$:

(i) General term :

(r + 1)th term of $(x+y)^n$ is $T_{r+1} = {}^nC_r x^{n-r} y^r$

(ii) Middle term/(s) :

(a) If n is even, there is only middle term, which is $\left(\frac{n+2}{2}\right)$ th term.

(b) if n is odd, there are two middle terms, which are

$\left(\frac{n+1}{2}\right)$ **th and** $\left(\frac{n+1}{2}+1\right)$ **th** terms.

(iii) Numerically greatest term in the expansion of $(x+y)^n$, n ∈ N

Let T_r and T_{r+1} be the rth and (r + 1)th terms respectively

$T_r = {}^nC_{r-1}\, x^{n-(r-1)}\, y^{r-1}$

$T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$

Now, $\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}_nC_{r-1}}\frac{x^{n-r}y^r}{x^{n-r+1}y^{r-1}}\right| = \frac{n-r+1}{r}.\left|\frac{y}{x}\right|$

Consider $\left|\frac{T_{r+1}}{T_r}\right| \geq 1$, $\left(\frac{n-r+1}{r}\right)\left|\frac{y}{x}\right| \geq 1$, $\frac{n+1}{r} - 1 \geq \left|\frac{x}{y}\right|$, $r \leq \frac{n+1}{1+\left|\frac{x}{y}\right|}$

Practice Problems :

1. **Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1\cdot3\cdot5...(2n-1)}{n!}\cdot 2^n\cdot x^n$, where $n \in N$.**
2. **Show that the coefficient of the middle term in the expansion of $(1 + x)^{2n}$ is the sum of the coefficients of two middle terms in the expansion of $(1 + x)^{2n-1}$.**
3. **Find the value of r, if the coefficients of (2r + 4)th and (r – 2)th terms in the expansion of $(1 + x)^{18}$ are equal.**
4. **If the coefficient of (r – 1)th, rth and (r + 1)th terms in the expansion of $(x + 1)^n$ are in the ratio 1 : 3 : 5, find n and r**
5. **The 2nd, 3rd and 4th terms in the expansion of $(x + y)^n$ are 240, 720 and 1080 respectively. Find the values of x, y and n.**
6. **Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 – x)^7$ using binomial theorem.**
7. **Find the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$.**
8. **Find the coefficient of a^4 in the product $(1 + 2a)^4 (2 – a)^5$ using binomial theorem.**
9. **The sum of the coefficients of the first three terms in the expansion of $\left(x - \frac{3}{x^2}\right)^m, x \neq 0$, m being a natural number, is 559. Find the term of the expansion containing x^3.**
10. **Show that the greatest coefficients in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1.3.5....(2n-1).2^n}{n!}$.**
11. **Express $(x+\sqrt{x^2+1})^6 + (x-\sqrt{x^2+1})^6$ as a polynomial in x.**
12. **If a_1, a_2, a_3 and a_4 be any four consecutive coefficients in the expansion of $(1 + x)^n$, prove that**

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}.$$

[Answers : (3) 6 (4) n = 7, r = 3 (5) x = 2, y = 3 and n = 5 (6) 171 (7) 5/12 (8) –438 (9) –5940 x^3]

C5 Multinominal Theorem

As we know the Binomial Theorem $(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r}y^r = \sum_{r=0}^{n} \frac{n!}{(n-r)!r!} x^{n-r}y^r$

putting $n - r = r_1$, $r = r_2$

therefore, $(x+y)^n = \sum_{r_1+r_2=n} \frac{n!}{r_1!r_2!} x^{r_1}.y^{r_2}$

Total number of terms in the expansion of $(x + y)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 = n$ i.e. ${}^{n+2-1}C_{2-1} = {}^{n+1}C_1 = n + 1$

In the same fashion we can write the multinominal theorem

$$(x_1 + x_2 + x_3 + ...x_k)^n = \sum_{r_1+r_2+....+r_k=n} \frac{n!}{r_1!r_2!...r_k!} x_1^{r_1}.x_2^{r_2}....x_k^{r_k}$$

Here total number of terms in the expansion of $(x_1 + x_2 + ... + x_k)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 + ... + r_k = n$ i.e. ${}^{n+k-1}C_{k-1}$

Practice Problems :

1. **(i) the middle term in the expansion of $\left(x - \frac{1}{2y}\right)^{10}$ (ii) the coefficient of x^{32} and x^{-17} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$**

2. **Find the coefficient of x^5 in the expansion of the product $(1 + 2x)^5 (1 – x)^7$.**

[Answers : (1) (i) $-\frac{63x^5}{8y^5}$ (ii) 1365, – 1365 (2) 171]

C6 Properties of Binomial Coefficients :

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_rx^r + ... + C_nx^n$ ....(1)

(1) The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n

Putting x = 1 in (1)

${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$ ....(2)

or $$\sum_{r=0}^{n} {}^nC_r = 2^n$$

(2) Again putting x = –1 in (1), we get

${}^nC_0 – {}^nC_1 + {}^nC_2 – {}^nC_3 + ... + (–1)^n\, {}^nC_n = 0$ ....(3)

or $$\sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0$$

(3) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1} i.e.,

${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = 2^{n-1}$

${}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$

(4) Sum of two consecutive binomial coefficients

${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(5) Ratio of two consecutive binomial coefficients $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(6) $${}^nC_r = \frac{n}{r}\, {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\, {}^{n-2}C_{r-2}$$

Practice Problems :

1. Prove the following identities :

(a) $^nC_0 + {}^nC_2 + {}^nC_4 + ... = 2^{n-1}$

(b) $^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$

(c) $^nC_0 + 3\,{}^nC_1 + 5\,{}^nC_2 + ... + (2n + 1){}^nC_n = (n + 1)2^n$

(d) $^nC_1 - 2\,{}^nC_2 + 3\,{}^nC_3 - .... + (-1)^{n-1}\, n{}^nC_n = 0$

(e) $C_1 + 2\,C_2 + 3\,C_3 + ... + n\,C_n = n\,2^{n-1}$

(f) $C_0 + 2\,C_1 + 3\,C_2 + ... + (n + 1)\,C_n = 2^n + n\,2^{n-1}$

(g) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... = \frac{1}{n+1}$

(h) $2C_0 + 2^2\frac{C_1}{2} + 2^3\frac{C_2}{3} + ... + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

(i) $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + ... = \frac{2^n}{n+1}$

(j) $2C_0 + 5\,C_1 + 8\,C_2 + ... + (3n + 2)\,C_n = (3n + 4)\,2^{n-1}$

C7 Binomial Theorem For Negative Integer or Fractional Indices

If $n \in R$ then,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

$$+ \frac{n(n-1)(n-2).....(n-r+1)}{r!}x^f + ...\infty$$

Remarks

(i) The above expansion is valid for any rational number other then a whole number if $|x| < 1$.

(ii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1 + x)^n$ is infinite. and the symbol ^nC_r cannot be used to denote the coefficient of the general term.

(iii) The first terms must be unity in the expansion, when index 'n' is a negative integer or fraction.

(iv) The general term in the expansion of $(1 + x)^n$ is $T_{r+1} = \frac{n(n-1)(n-2).....(n-r+1)}{r!}x^r$

(v) When 'n' is any rational number other than whole number then approximate value of $(1 + x)^n$ is $1 + nx$ (x^2 and higher powers of x can be neglected)

(vi) Expansion to be remembered ($|x| < 1$)

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ... + (-1)^r x^r + ...\infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^5 + ... + x^r + ... \infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (-1)^r (r + 1) x^r + ... \infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... + (r + 1)x^r + ... \infty$

Practice Problems :

1. Find the coefficient of x^6 in the expansion of $(1 - 2x)^{-5/2}$.

2. Find the coefficient of x^{10} in the expansion of $\frac{(1+3x^2)}{(1-x^2)^3}$, mentioning the condition under which the result holds.

[Answers : (1) $\left[\frac{15015}{16}\right]$]