Rotational & Rolling Motion

ROTATIONAL & ROLLING MOTION

C1A **TYPES OF MOTION**

Motion of an object can be of three kinds :

(a) Translational Motion

(b) Rotational Motion

(c) Rolling Motion

C1B **ROTATIONAL MOTION**

Here we examine the rotation of a rigid body (a body with a definite and unchanging shape and size) about a fixed axis (an axis that does not move), called the axis of rotation or the rotational axis. Every point of the body moves in a circle whose centre lies on the axis of rotation, and every point moves through the same angle during a particular time. In pure translation, every point of the body moves through the same linear distance during a particular time interval in a straight line. Hence we can see the angular equivalent of the linear quantities position, displacement, velocity and acceleration.

Angular displacement : $\Delta\theta = \theta_2 - \theta_1$.

Angular velocity : Average angular velocity, $\langle\omega\rangle = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$.

Instant angular velocity, $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$.

Both $\langle\omega\rangle$ and ω are vectors, with the direction given by the right hand rule. The magnitude of the body's angular velocity is the angular speed.

Angular acceleration : Average angular acceleration, $\langle\alpha\rangle = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$.

Instant angular acceleration, $\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta}$.

Both $\langle\alpha\rangle$ and α are vectors.

Practice Problems :

1. The angular position of a reference line on a spinning wheel is given by $\theta = t^3 - 27t + 4$, where t is in seconds and θ is in radians. Find angular speed and angular acceleration.

2. A child's top is spun with angular acceleration $\alpha = 5t^3 - 4t$ where the coefficients are in units compatible with seconds and radians. At t = 0, the top has angular velocity 5 rad/s, and a reference line on it is at angular position $\theta = 2$ rad. Find angular velocity and angular position of the top.

C1C **ROTATIONAL MOTION WITH CONSTANT ANGULAR ACCELERATION**

The kinematics equations for constant angular acceleration

$\omega = \omega_0 + \alpha(t - t_0)$

$\theta = \theta_0 + \omega_0(t - t_0) + (1/2)\alpha\,(t - t_0)2$

$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$. Here the symbols have the following meaning :

$\theta_0 \rightarrow$ Angular positon at t_0

$\theta \rightarrow$ Angular position at t

$\omega_0 \rightarrow$ Angular velocity at t_0

$\omega \rightarrow$ Angular velocity at t

$\alpha \rightarrow$ Angular acceleration.

Practice Problems :

1. **A wheel is making revolutions about its axis with uniform angular acceleration. Starting from rest, it reaches 100 rev/sec in 4 seconds. Find the angular acceleration. Find the angle rotated during these four seconds.**
2. **A wheel rotating with uniform angular acceleration covers 50 revolutions in the first five seconds after the start. Find the angular acceleration and the angular velocity at the end of five seconds.**
3. **A wheel starting from rest is uniformly accelerated at 4 rad/s^2 for 10 seconds. It is allowed to rotate uniformly for the next 10 seconds and is finally brought to rest in the next 10 seconds. Find the total angle rotated by the wheel.**
4. **A body rotates about a fixed axis with an angular acceleration of one radian/$second^2$. Through what angle does it rotate during the time in which its angular velocity increases from 5 rad/s to 15 rad/s.**
5. **Find the angular velocity of a body rotating with an acceleration of 2 rev/s^2 as it completes the 5th revolution after the start.**
6. **The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds. (i) What is its angular acceleration, assuming the acceleration to be uniform ? (ii) How many revolutions does the engine make during this time ?**

[Answers : (1) 25 rev/s^2, 400 π rad (2) 4 rev/s^2, 20 rev/s (3) 800 rad (4) 100 rad (5) $2\sqrt{5}$ rev/s (6) (i) 4π rad/s^2 (ii) 576]

C1D RELATION BETWEEN LINEAR AND ANGULAR VARIABLES

A point in a rigid rotating body at a perpendicular distance r from the rotation axis moves in a circle with radius r. If the body rotates through an angle θ, the point moves along the arc with length s is given by $s = \theta r$ where θ is in radians.

The linear velocity $\vec{v}$ of the point is tangent to the circle and the point's linear speed v is given by $v = \omega r$, where ω is the angular speed of the body.

The linear acceleration $\vec{a}$ of the point has both tangential and radial components. The tangential component is $a_t = \alpha r$, where α is the magnitude of the angular acceleration of the body. The tangential acceleration $\left(a_t = \frac{dv}{dt}\right)$ represents only the part of linear acceleration that is responsible for change in the magnitude of the linear velocity $\vec{v}$. Like $\vec{v}$, that part of the linear acceleration is tangent to the path of the point in question.

The radial component, responsible to change the direction of $\vec{v}$, is $a_r = \frac{v^2}{r} = \omega^2 r$. This component is directed radially inward.

For a body having constant velocity, $a_t = 0$ and $a_r = 0$. For a body having constant angular speed or linear speed, $a_t = 0$ and $a_r = \frac{v^2}{r}$. Body having variable angular speed or linear speed, $a_t = \frac{dv}{dt}$ and $a_r = \frac{v^2}{r}$.

Remember the following vector relation

$$\vec{v} = \vec{\omega} \times \vec{r}$$
$$\vec{a} = \vec{a}_t + \vec{a}_r$$
$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Practice Problems :

1. A disc of radius 10 cm is rotating about its axis at an angular speed of 20 rad/s. Find the linear speed of
   (a) a point on the rim,
   (b) the middle point of a radius.
2. A disc rotates about its axis with a constant angular acceleration of 4 rad/s^2. Find the radial and tangential acceleration of a particle at a distance of 1 cm from the axis at the end of the first second after the disc starts rotating.

[Answers : (1) 2 m/s, 1 m/s (2) 16 cm/s^2, 4 cm/s^2]

C2A ROTATIONAL KINETIC ENERGY AND ROTATIONAL INERTIA

Consider a body (rotating with angular velocity ω) as being made up a large number of particles with masses m_1, m_2,....at distances r_1, r_2,....from the axis of rotation. The total kinetic energy of the body is given by

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ...$$
$$= \sum_{i=1} \frac{1}{2}m_iv_i^2 = \sum_{i=1} \frac{1}{2}m_i(\omega r_i)^2$$
$$= \sum_{i=1} \frac{1}{2}m_ir_i^2\omega^2 = \frac{1}{2}\left(\sum m_ir_i^2\right)\omega^2$$

The quantity in parantheses on the right side of the above equation tells us how the mass of the rotating body is distributed about its axis of rotation. We call this quantity the rotational inertia (or moment of inertia) I of the body with respect to the axis of rotation. In terms of moment of inertia I, the rotational kinetic energy K of a rigid body is $K = \frac{1}{2}I\omega^2$.

Remember the following :

Moment of inertia of point mass I = mr^2

Moment of inertia of a system of discret particles $I = \sum_{i=1} m_ir_i^2$.

Moment of inertia of a system for constinuously mass distribution $I = \int r^2 dm$

Here r and r_i in these expressions represents the perpendicular distance from the axis of rotation of each mass element in the body.

This quantity has the same significance in rotational motion as that of mass in linear motion. It is a measure of the resistance offered by a body to a change in its rotational motion.

C2B THEOREM OF MOMENT OF INERTIA :

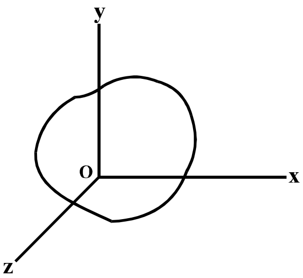
1. The parallel axis theorem

The parallel axis theorem relates the moment of inertia I of a body about any axis to that of the same body about a parallel axis through the centre of mass (also known as centroid axis) as $I = I_{com} + M h^2$ where I_{com} is the moment of inertia of the body about the centroidal axis and h is the perpendicular distance between the two axes. Note that the parallel axis may lie within or inside the body.

2. The perpendicular axis theorem

This theorem is valid for a planar or laminar body (body in two dimensions like a thin disc, ring or thin plate etc.).

Let x and y be the two axes which lie in the plane of the body and pass through the point O, as shown in figure.

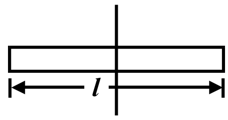


Then the moment of inertia about an axis (called z-axis) passing through O and perpendicular to the plane containing x and y axes is given by $I_z = I_x + I_y$, where I_x, I_y and I_z are the respective moment of inertia of the body about x, y and z axes.

C2C MOMENT OF INERTIA OF IMPOTANT BODIES :

1. A thin rod

(a) About an axis through centre of mass perpendicular to length, $I = \frac{1}{12}Ml^2$



(b) About an axis through one of the end and perpendicular to length, $I = \frac{1}{3}ML^2$

2. A ring or Hoop

(a) About an central axis and perpendicular to the plane $I = MR^2$

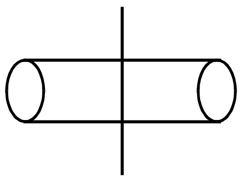
(b) About any diameter $I = \frac{MR^2}{2}$

(c) About a tangent in the plane of ring $I = \frac{3}{2}MR^2$

(d) About a tangent perpendicular to the plane of ring $I = 2\,MR^2$

3. Hollow cylinder

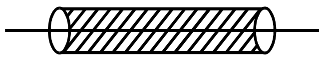
About central axis $I = MR^2$



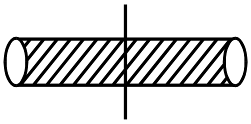
$$I = \frac{ML^2}{12} + \frac{MR^2}{2}$$

4. Solid cylinder

(a) **About central axis or axis of symmetry** $I = \frac{MR^2}{2}$



(b) **About central diameter** $I = \frac{ML^2}{12} + \frac{MR^2}{4}$



5. Disc

(a) **About an axis through the centre and perpendicular to the plane** $I = \frac{MR^2}{2}$.

(b) **About the diameter** $I = \frac{MR^2}{4}$

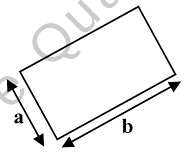
(c) **About the tangent perpendicular to the plane of disc** $I = \frac{3}{2}MR^2$

(d) **About the tangent in the plane of disc** $I = \frac{5}{4}MR^2$

6. A rectangular plate

About perpendicular axis through centre

$$I = \frac{1}{12}M(a^2 + b^2)$$



7. Solid sphere

(a) **About its diameter** $I = \frac{2}{5}MR^2$

(b) **About its tangent** $I = \frac{7}{5}MR^2$

8. Hollow sphere

(a) **About its diameter** $I = \frac{2}{3}MR^2$

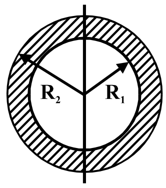
(b) **About its tangent** $I = \frac{5}{3}MR^2$

9. Annular cylinder (or ring)

About central axis $I = \frac{1}{2}M(R_1^2 + R_2^2)$ **, here R_2 is the outer radius and R_1 is the inner radius.**

10. A spherical shell

About the diameter $I = \frac{2}{5} M \frac{(R_2^5 - R_1^5)}{(R_2^3 - R_1^3)}$.



<u>Practice Problems :</u>

1. The moment of inertia of a body about an axis is 1.2 kg-m^2. Initially the body is at rest. In order to produce a rotational kinetic energy of 1500 J, an angular acceleration of 25 rad/s^2 must be applied about the axis for a duration of

(a) 2 s (b) 4 s (c) 8 s (d) 10 s

2. Which of the following has the highest moment of inertia if each has the same mass and the same radius ?

(a) A ring about its axis perpendicular to the plane of the ring

(b) A solid sphere about one of its diameters

(c) A spherical shell about one of its diameters

(d) A disc about its axis perpendicular to the plane of its disc.

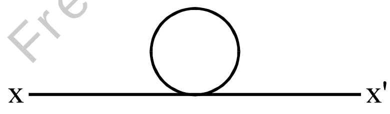
3. The moment of inertia of a uniform circular disc about a diameter is *I*. Its momentum of inertia about an axis perpendicular to its plane and passing through a point on its rim is

(a) 3 *I* (b) 4 *I* (c) 5 *I* (d) 6 *I*

4. The radius of gyration of a rod of mass m and length L about an axis of rotation perpendicular to its length and passing through the center is

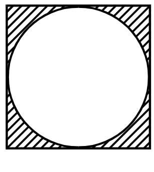
(a) $\frac{L}{2\sqrt{3}}$ (b) $\frac{L}{2\sqrt{2}}$ (c) $\frac{L}{2\sqrt{5}}$ (d) $\frac{L}{\sqrt{2}}$

5. A wire of mass per unit length λ and length L is used to form a circular loop. The moment of inertia about the xx' is



(a) $\frac{2\lambda L^3}{8\pi^2}$ (b) $\frac{4\lambda L^3}{8\pi^2}$ (c) $\frac{3\lambda L^3}{8\pi^2}$ (d) $\frac{5\lambda L^3}{8\pi^2}$

6. From a uniform square plate of mass M and length L, a circular plate is removed and the remaining part is shown in figure

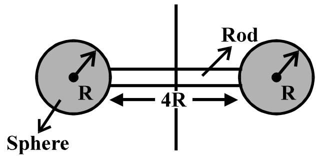


The moment of inertia of the remaining part passing through the centre and perpendicular to the plane is

(a) $ML^2\left(\frac{1}{6}-\frac{\pi}{32}\right)$ (b) $ML^2\left(\frac{1}{3}-\frac{\pi}{32}\right)$

(c) $ML^2\left(\frac{1}{2}-\frac{\pi}{16}\right)$ (d) $ML^2\left(\frac{1}{2}-\frac{\pi}{64}\right)$

7. The mass of each sphere are M and the mass of the rod is M. The moment of inertia of the given figure about the axis shown in figure is



(a) $\frac{301MR^2}{15}$ (b) $\frac{302MR^2}{15}$ (c) $\frac{303MR^2}{15}$ (d) $\frac{304MR^2}{15}$

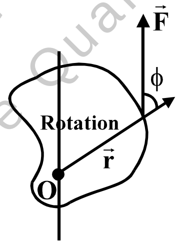
8. Two loops P and Q are made from a uniform wire. The radii of P and Q are r_1 and r_2 respectively, and their moments of inertia are I_1 and I_2 respectively. If $I_2/I_1 = 4$ then $\frac{r_2}{r_1}$ equals

(a) $4^{2/3}$ (b) $4^{1/3}$ (c) $4^{-2/3}$ (d) $4^{-1/3}$

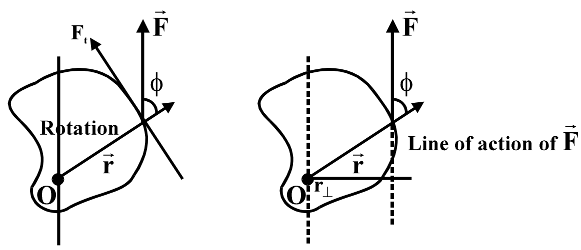
[Answers : (1) a (2) a (3) d (4) a (5) c (6) a (7) b (8) b]

C3 MOMENT OF FORCE OR TORQUE

Torque is a turning or twisting action on a body about a rotation axis due to a force $\vec{F}$. It has the same role in rotational motion as that of force in linear motion. Consider a force $\vec{F}$ is exerted at a point given by the position vector $\vec{r}$ relative to the axis, as shown in figure. Its torque about O is given by $\vec{\tau} = \vec{r} \times \vec{F}$.



Note that torque is always defined with reference to a specific point, often (but not always) the origin of a co-ordinate system.



The magnitude of the torque is $\tau = rF_t = r_\perp F = rF\sin\phi$ where F_t is the component of $\vec{F}$ perpendicular to $\vec{r}$ and ϕ is the angle between $\vec{r}$ and $\vec{F}$.

The quantity $r_\perp$ is the perpendicular distance between the rotation axis and extended line runnning through the $\vec{F}$ vector. This line is called the line of action of $\vec{F}$ and $r_\perp$ is called the moment arm of $\vec{F}$. Similarly r is the moment arm of F_t.

Newton's Second Law for Rotation

The rotational analog of newton's second law is $\tau_{net} = I\alpha$

where τ_{net} is the net torque acting on a particle or rigid body, I is the rotational inertia of the particle or body about the rotation axis, and α is the resulting angular acceleration about that axis. Here τ_{net} and I are taken with respect to the same rotation axis.

Note the following points for $\tau_{net} = I\,\alpha$

1. This equation is valid only for rigid bodies.
2. Here we consider the torque of the external forces.
3. The axis of rotation should be stationary. But, infact, this equation is valid even when the axis of rotation moves if the following two conditions are met :
   (a) The moving axis of rotation must be an axis of symmetry.
   (b) The axis must not change direction.

Equilibrium : A rigid body is said to be in equilibrium if

(a) Net external force equal to zero. This is the condition of transiational equilibrium $\sum \vec{F} = 0$.

(b) Net external torque equal to zero. This is the condition of rotational equilibrium $\sum \vec{\tau} = 0$.

<u>Practice Problems :</u>

1. A wheel of radius 10 cm and mass 12.5 kg rotates freely about an axis passing through the center and perpendicular to the plane of the wheel by applying a constant force F and it is found that its angular speed increases from zero to 2 rad/s in 1s. The force F acting on the wheel to do so

   (a) 1.25 N (b) 2.5 N (c) 4.5 N (d) 6.25 N

2. A cubical block of mass m and edge length *l* slides down the rough inclined plane of inclination α with a uniform velocity. (a) Draw the force body diagram of cubical block showin all the forces and its point of application. (b) What is the torque of the normal force acting on the block about its centre.

3. A rod of mass m and length L, lying horizontally, is free to rotate about a vertical axis through its centre. A horizontal force of variable magnitude F equal to αt acts on the rod at a distance of L/4 from the centre. The force is always perpendicular to the rod. Find the angle rotated by the rod during the time t after the motion starts.

4. A square plate of mass m and edge length *l* rotates about one of the diagonal of plate with uniform angular acceleration α by an external agent. What is torque applied by the external agent ?

5. A flywheel of moment of inertia 5.0 kg-m^2 is rotated at a speed of 60 rad/s. Because of the friction at the axle, it comes to rest in 5.0 minutes., Find (a), the average torque of the friction, (b) the total work done by the friction and (c) the angular momentum of the wheel 1 minute before it stops rotating.

6. A light rod of length 1 m is pivoted at its centre and two masses of 5 kg and 2 kg are hung from the ends. Find the initial angular acceleration of the rod assuming that it was horizontal in the beginning.

7. Two blocks of masses m and M connected by a string passing over a pulley. The horizontal table over which the mass m slides is smooth. The pulley has a radius r and moment of inertia I about its axis and it can freely rotate about this axis. Find the acceleration of the mass M assuming that the string does not slip on the pulley.

8. A uniform metre stick of mass 200 g is suspended from the ceiling through two vertical strings of equal lengths fixed at the ends. A small object of mass 20 g is placed on the stick at a distance of 70 cm from the left end. Find the tensions in the two strings.

9. **A uniform ladder of length 10.0 m and mass 16.0 kg is resting against a vertical wall making an angle of 37^0 with it. The vertical wall is frictionless but the ground is rough. An electricial weighing 60.0 kg climbs up the ladder. If he stays on the ladder at a point 8.00 m from the lower end, what will be the normal force and the froce of friction on the ladder by the ground ? What should be the minimum coefficient of friction for the electricial to work safely ?**

10. **Suppose the friction coefficient between the ground and the ladder of the previous problem is 0.540. Find the maximum weight of a mechanic who could go up and do the work from the same position of the ladder.**

11. **A 6.5 m long ladder rests against a vertical wall reaching a height of 6.0 m. A 60 kg man stands half way up the ladder. (a) Find the torque of the force exerted by the man on the ladder about the upper end of the ladder. (b) Assuming the weight of the ladder to be negligible as compared to the man and assuming the wall to be smooth, find the force exerted by the ground on the ladder.**

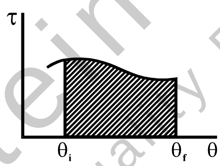
12. **The door of an almirah is 6 ft high, 1.5 ft wide and weighs 8 kg. The door is supported by two hinges situated at a distance of 1 ft from the ends. If the magnitudes of the forces exerted by the hinges on the door are equal, find this magnitude.**

[Answers : (1) a (2) ½mg*l* sinα (5) (a) 1.0 N-m (b) 9.0 kJ (c) 60 kg-m²/s (6) 8.4 rad/s² (7) $\frac{Mg}{M+m+I/r^2}$ (8) 1.04 N in the left string and 1.12 N in the right (9) 745 N, 412 N, 0.553 (10) 44.0 kg (11) (a) 740 N-m (b) 590 N vertical and 120 N horizontal (12) 43 N]

C4 WORK IN ROTATIONAL MOTION

When a torque τ acts on a rigid body that undergoes an angular displacement from θ_i to θ_f then work W done by the torque is $W = \int_{\theta_i}^{\theta_f} \tau d\theta$. If the torque is constant, then $W = \tau(\theta_f - \theta_i) = \tau\Delta\theta$.

Graphical interpreration of rotational work done is shown in figure.



Work - Energy Theorem for Rotational Motion

Work energy theorem for rotational motion of a rigid body is $W = \Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$.

Practice Problems :

1. **A uniform cylinder of radius R and mass M is spinned about its axis to the angular velocity ω_0 and then placed into a corner. The coefficient of friction between the walls and the cylinder is equal to μ. The total work done is**

(a) $-\frac{1}{2}MR^2\omega_0^2$ (b) $\frac{1}{2}MR^2\omega_0^2$ (c) $\frac{1}{4}MR^2\omega_0^2$ (d) $-\frac{1}{4}MR^2\omega_0^2$

[Answers : (1) d]

C5 POWER IN ROTATIONAL MOTION

When the body rotates with angular velocity ω, the power P (rate at which the torque does work) is $P = \frac{dW}{dt} = \tau\omega$.

Practice Problems :

1. An electric motor exerts a constant torque of $\tau = 10$ N-m on a gridstone mounted on it shaft. The moment of inertia of the grindstone is 2 kg-m^2. If the system starts from rest, the kinetic energy at the end of 8s is

(a) 400 J (b) 800 J (c) 1600 J (d) 2000 J

2. In the above problem, the instant power at t = 8s delivered by the motor is

(a) 100 W (b) 200 W (c) 400 W (d) 800 W

[Answers : (1) c (2) c]

C6 ANGULAR MOMENTUM

1. **Angular momentum of a particle**

The angular momentum $\vec{L}$ of a particle, with linear momentum $\vec{p}$, mass m and linear velocity $\vec{v}$ is a vector quantity defined relative to a fixed point (usually an origin). It is $\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$.

The magnitude of $\vec{L}$ is given by

$|\vec{L}| = mvr \sin\phi = rp_{\perp} = rmv_{\perp} = r_{\perp}p = r_{\perp}mv$.

where ϕ is the angle between $\vec{r}$ and $\vec{p}$, $p_{\perp}$ and $v_{\perp}$ are the components of $\vec{p}$ and $\vec{v}$ respectively, perpendicular to $\vec{r}$ and $r_{\perp}$ is the perpendicular distance between the fixed point and the line of extension of $\vec{p}$. The direction of $\vec{L}$ is given by the right hand rule for cross products.

2. **Angular momentum of a system of particles**

The angular momentum $\vec{L}$ of a system of particles is the vector sum of the angular momentum of individual particles, $\vec{L} = \sum_{i=1}^{n} \vec{L}_i$.

3. **Angular Momentum of a Rigid Body**

When a symmetric rigid body with moment of inertia I rotates with angular velocity $\vec{\omega}$ about a stationary axis of symmetry, its angular momentum is given by $\vec{L} = I\vec{\omega}$. If the body is not symmetric or the rotation axis is not an axis of symmetry, the component of angular momentum along the axis of rotation is equal to Iω.

Practice Problems :

1. A particle of mass m is projected with speed u at an angle of θ with the horizontal in the vertical plane. Find the angular momentum of the particle about the point of projection (a) when the particle at at the highest point (b) just before the landing.

2. Two spheres each of radius r and mass m are connected by a rod of mass m and length 2r. The system is rotated with angular speed ω about an axis passing through the mid-point of the rod and perpendicular to the length of the rod. Find the angular momentum about this axis of this system.

3. A mass M is moving with a constant velocity parallel to the x-axis. Its angular momentum with respect to the origin

(a) is zero (b) remains constant

(c) goes on increasing (d) goes on decreasing

4. When a mass is rotated in a plane about a fixed point, its angular momentum is directed along

(a) the radius

(b) the tangent to the orbit

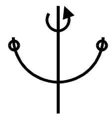
(c) a line at an angle of 45^0 to the plane of rotation

(d) the axis of rotation.

5. If the radius of earth contarcts to half of its present day value, the mass remaining unchanged, the duration of the day will be

(a) 48 hrs (b) 24 hrs (c) 12 hrs (d) 6 hrs

6. Two beads (each of mass m) can move freely in a frictionless semicircular wire of mass m and radius is r. The angular velocity ω_0 when the beads are at distance r from the axis as shown in figure. The angular velocity of the system when the beads are at a distance r/2 from the axis is



(a) $\frac{5}{4}\omega_0$ (b) $\frac{3}{2}\omega_0$ (c) $\frac{5}{2}\omega_0$ (d) $2\,\omega_0$

[Answers : (3) b (4) d (5) d (6) c]

C7 RELATION BETWEEN TORQUE AND ANGULAR MOMENTUM

The rate of change of angular momentum of a rigid body equals the net torque acting on it i.e., $\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$.

Practice Problems :

1. A constant torque acting on a uniform circular wheel changes its angular momentum from A_0 to $4A_0$ in 4 seconds. The magnitude of this torque is

(a) $\frac{3A_0}{4}$ (b) A_0 (c) $4\,A_0$ (d) $12\,A_0$

[Answers : (1) a]

C8 CONSERVATION OF ANGULAR MOMENTUM

The angular momentum $\vec{L}$ of a system remains constant if the net external torque acting on the system is zero i.e. $\vec{\tau} = \frac{d\vec{L}}{dt} = 0 = \vec{L} = \text{constant}$.

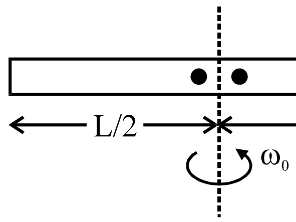
This is a law of conservation of angular momentum. It is one of the fundamental conservation laws of nature, having been verified even in situation (involving high speed particles or subatomic dimension) in which newton's laws are not applicable.

Practice Problems :

1. A thin circular ring of mass M is rotating about its axis with a constant angular velocity ω. Two objects, each of mass m, are attached gently to the opposite ends of a diameter of the ring. The ring now rotates with an angular velocity.

(a) $\frac{\omega M}{M+m}$ (b) $\frac{\omega(M-2m)}{M+2m}$ (c) $\frac{\omega M}{M+2m}$ (d) $\frac{\omega(M+2m)}{M}$

2. A smooth uniform rod of length L and mass M has two identical beads of negligible size, each of mass m, which can slide freely along the rod. Initially the two beads are at the centre of the rod and the system is rotating with an angular velocity ω_0 about an axis perpendicular to the rod and passing through the mid-point of the rod. There are no external forces. When the beads reach the ends of the rod, the angular velocity of the system is



(a) $\frac{M\omega_0}{M+3m}$ **(b)** $\frac{M\omega_0}{M+4m}$ **(c)** $\frac{M\omega_0}{M+5m}$ **(d)** $\frac{M\omega_0}{M+6m}$

[Answers : (1) c (2) d]

C9 **ANGULAR IMPULSE**

The angular impulse of a torque τ in a time interval dt is defined as $d\vec{J} = \vec{\tau}dt \Rightarrow \Delta\vec{J} = \int_{t_1}^{t_2} \vec{\tau}dt$.

$\therefore$ $\vec{\tau} = \frac{d\vec{L}}{dt}$

$\therefore$ $\Delta\vec{J} = \Delta\vec{L} = \vec{\tau}\Delta t$

Thus, the change in angular momentum is equal to the angular impulse of the resultant torque. This theorem is known as angular impulse - momentum theorem.

Practice Problems :

1. **A uniform rod of mass M and length L is placed on a frictionless horizontal surface with one of the end of the rod is tapped. A sharp linear impulse J, perpendicular to length of the rod, is provided at the mid-point of the rod. The time taken by the rod to complete one revolution is**

(a) $\frac{4ML\pi}{3J}$ **(b)** $\frac{\pi ML}{12J}$ **(c)** $\frac{2ML\pi}{3J}$ **(d)** $\frac{\pi ML}{6J}$

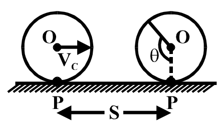
[Answers : (1) a]

C10 **ROLLING MOTION**

Rigid body rotation about a moving axis i.e., the motion of the body as combined translation and rotation is defined as rolling motion. The key to understanding such situation is this : Every possible motion of a rigid body can be represented as a combination of translational motion of the centre of mass and rotation about an axis through the centre of mass. Note that the axis passing through the centre of mass is a symmetry axis that does not change direction as the body moves.

Rolling without slipping (Pure Rolling)

In case of pure rolling the point P at which the body makes contact with the surface over which the whell rolls move same distance as the centre of mass O of the rolling body during the same time i.e. s = θ R



where R is the radius of rolling body.

Differentiating the above equation w.r.t. time, we get $v_c = R\omega$.

Further differentiation w.r.t. to time gives $a_c = R\alpha$.

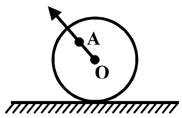
Note that $v_c = \omega R$ is not the condition of pure rolling. Condition of pure rolling; The instantaneous velocity of the point of contact.

C11 <u>DISPLACEMENT, VELOCITY AND ACCELERATION OF A POINT ON THE ROLING BODY</u>

Consider a point A on the rolling body. Here O is centre of mass of the body.

Displacement of A

$\Delta\vec{r}_A = \Delta\vec{r}_0 + \Delta\vec{r}_{AO}$, here $\Delta\vec{r}_{AO}$ is the displacement of point A w.r.t. O.



Velocity of A

$\vec{v}_A = \vec{v}_0 + \vec{v}_{AO}$, here $\vec{v}_{AO}$ is the velocity of point A w.r.t. O which equals to $\vec{v}_{AO} = \vec{\omega}\times\vec{r} = \vec{\omega}\times\overrightarrow{OA}$ where $\vec{\omega}$ is the angular velocity vector normal to the plane of the motion in the sense determined by the right hand rule.

Acceleration of A

$\vec{a}_A = \vec{a}_0 + \vec{a}_{AO}$, here $\vec{a}_{AO}$, acceleration of A w.r.t. O, has two components $\vec{a}_{AO} = (\vec{a}_{AO})_n + (\vec{a}_{AO})_t$. Here

$(\vec{a}_{AO})_n$ = normal component of $\vec{a}_{AO} = \vec{\omega}\times(\vec{\omega}\times\vec{r}) = \vec{\omega}\times(\vec{\omega}\times\overrightarrow{OA}) = \vec{\omega}\times\vec{v}_{AO}$.

The magnitude of $(\vec{a}_{AO})_n$

$$|(\vec{a}_{AO})_n| = \omega^2 r = \frac{v_{AO}^0}{r}$$

And $(\vec{a}_{AO})$ = tangential component of $\vec{a}_{AO} = \vec{\alpha}\times\vec{r} = \vec{\alpha}\times\overrightarrow{OA}$

Here $\vec{\alpha}$ is the angular acceleration of the body.

C12 <u>ROLLING AS PURE ROTATION</u>

For a body in pure rolling, the point of contact P has the zero instantaneous velocity. Hence an axis passing through the point P and perpendicular to the plane of body is an axis of rotation or the rotation axis and about point P we can consider the pure rotation of the pure rolling body. This point P is called the instantaneous centre of rotation or instantaneous centre of zero velocity. This point may lie within or outside the body. Note that the angular velocity of the body about this point is the same as that about its centre of mass.

C13 <u>THE DYNAMICS OF ROLLING MOTION</u>

(a) For translation of centre of mass $\sum \vec{F}_{ext} = M\vec{a}_{CM}$

For rotation about the centre of mass $\sum \tau = I_{CM}\alpha$

(b) The work energy theorem for rolling

$$W_T + W_R = \Delta K = \frac{1}{2}MV_{2CM}^2 + \frac{1}{2}I_{cm}\omega_2^2 - \frac{1}{2}MV_{1CM}^2 - \frac{1}{2}I_{cm}\omega_2^2$$

C14 <u>ANGULAR MOMENTUM FOR ROLLING BODY</u>

$$\vec{L} = \vec{L}_{CM} + M\vec{r}\times\vec{v}_{CM}$$

The first term $\vec{L}_{CM}$ represents the angular momentum of the body about the centre of mass frame. The second term $M\vec{r}\times\vec{v}_{CM}$ equals the angular momentum of the body about the particular point if the body is assumed to be concentrated at the centre of mass translating with the velocity $\vec{v}_{cm}$.

Practice Problems :

1. A solid cylinder of mass M and radius R rolls down an inclined plane from height h without slipping. The speed of its centre of mass when it reaches the bottom is

(a) $\sqrt{2gh}$ (b) $\sqrt{\frac{4}{3}gh}$ (c) $\sqrt{\frac{3}{4}gh}$ (d) $\sqrt{\frac{4g}{h}}$

2. A thin, uniform, circular disc is rolling down an inclined plane of inclination 30^0 without slipping. Its linear acceleration along the plane is

(a) g/4 (b) g/3 (c) g/2 (d) 2g/3

3. A solid sphere, a hollow sphere and a solid cylinder, all of the same radius, roll down an inclined plane from the same height, starting from rest. Which of them takes the least time in reaching the bottom of the plane ?

(a) Solid sphere (b) Hollow sphere
(c) Solid cylinder (d) All will take the same time

4. A ring is rolling without slipping on a horizontal surface. The velocity of centre of mass of the ring is v. The fraction of rotational kinetic energy of the total kinetic energy is

(a) 1/2 (b) 1/3 (c) 1/4 (d) 1/5

5. Find the angular momentum of the rolling disc having centre of mass speed v about any point on the horizontal surface ? Also find the kinetic energy of this body. What fraction of kinetic energy associated with the rotational part ? Do this problem for any rigid body which can roll on the horizontal surface. This body has the moment of inertia I.

6. A rigid body is rolling with slipping along the horizontal frictional surface. Which of the following quantity is conserved ?

(a) Linear momentum
(b) Angular momentum about the centre of mass
(c) Angular momentum about any point on the surface
(d) Kinetic energy

7. A sphere is rolling without slipping on a rough horizotnal surface. It collides with a smooth vertical wall. Which of the following statement is correct ?

(a) After the collision the sphere doesnot roll with slipping.
(b) After the collision the sphere has same angular speed before the collision.
(c) The linear velocity of the sphere after the collision doesnot change
(d) all are correct

[Answers : (1) b (2) b (3) a (4) a (6) c (7) b]