SEQUENCE AND SERIES

Syllabus :

Arithmetic and Geometric progression, insertion of arithmetic, geometric means between two given numbers, Relation between A.M. and G.M. Sum upto n terms of special series : S_n, S_{n^2}, S_{n^3}. Arithmetico - Geometric progression.

C1 Arithmetic Sequence or Arithmetic Progression

An Arithmetic sequence is a sequence in which the difference between any term and its just preceding term is constant throughout. This constant is called the common difference.

nth term of A.P. : $T_n = a + (n-1)d$

Sum of n terms of an A.P. : $S_n = \frac{n}{2}[2a + (n-1)d]$

If l is last term (or nth term), then $S_n = \frac{n}{2}\left[a + \underbrace{a + (n-1)d}_{l}\right] \Rightarrow S_n = \frac{n}{2}[a + l]$

Properties of an A.P. :

1. If a_1, a_2, a_3,..............are in A.P.
   Then $a_1 \pm k, a_2 \pm k, a_3 \pm k$, ... are in A.P.
2. If a_1, a_2, a_3,........................are in A.P.
   Then ka_1, ka_2, ka_3,.....................are in A.P.
   & $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}$ .... are in A.P.
3. If a_1, a_2, a_3,.....................are in A.P. & b_1, b_2, b_3 are in A.P.
   Then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3$,.................are in A.P.
4. If a_1, a_2, a_3,............................are in A.P. & b_1, b_2, b_3 are in A.P.
   Then a_1b_1, a_2b_2, a_3b_3,......................are not in A.P.
5. If a_1, a_2, a_3,.......................an are in A.P.
   Then sum of the terms equidistant from the beginning and end is constant
   i.e. $a_k + a_{n-(k-1)} = a_1 + a_n$ (first term + last term)
6. If n^{th} term of any sequence is a linear expression in n, then sequence is an A.P.
   If n^{th} term of form An + B, then common difference of A.P. is A.
7. If sum of n terms of any sequence is quadratic in n, then sequence is A.P.
   If the sum of n term is in the form $An^2 + Bn + C$, then common difference of A.P. is 2A.

Selection of Terms in an A.P.

No. of terms is odd	**Choosen terms**	**Common Differences**	**No. of terms is even**	**Choosen terms**	**Common Difference**
3	a – d, a, a + d	d	4	a – 3d, a – d, a + d, a + 3d	2d
5.	a – 2d, a – d, a, a + d a + 2d	d	6	a – 5d, a – 3d, a– d, a + d, a + 3d, a + 5d	2d

Practice Problems :

1. **The least value of 'a' for which $5^{1+x} + 5^{1-x}$, $\frac{a}{2}$, $25^x + 25^{-x}$ are three consecutive terms of an A.P. is**

   (a) 5 (b) 10 (c) 12 (d) none of these

2. **The first, second and middle terms of an A.P. are a, b, c respectively. Their sum is**

   (a) $\frac{2(c-a)}{b-a}$ (b) $\frac{2c(c-a)}{b-a}+c$ (c) $\frac{2c(b-a)}{c-a}$ (d) $\frac{2b(c-a)}{b-a}$

3. **The sum of first p terms of an A.P. is q and the sum of the first q terms is p. The sum of the first (p + q) terms is**

   (a) p + q (b) 0 (c) – (p + q) (d) – 2 (p + q)

4. **If the sum of first p terms, first q terms and first r terms of an A.P. be a, b and c respectively, then**

   $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)$ **is equal to**

   (a) 0 (b) 2 (c) pqr (d) $\frac{8abc}{pqr}$

[Answers : (1) c (2) b (3) c (4) a]

C2 Geometric Sequence or Geometric Progression :

A Geometric progression (G.P.) is a sequence in which the ratio of any term and its just preceding term is constant throughout.

This constant is called the common ratio.

Example : (i) $8, 1, 8^{-1}, 8^{-2}$,...........

(ii) 2, 4, 8, 16,.......

n^{th} Term of G.P.

If a is first term and r is the common ratio of geometric progression then n^{th} term

$$t_n = ar^{n-1}$$

Sum of First n Terms : $S_n = \frac{a(r^n-1)}{r-1}, r \neq 1$

If r < 1 then S_∞ (sum of infinite terms) = $\frac{a}{1-r}$ $(|r|<1)$

Properties of a G.P. :

1. If a_1, a_2, a_3,................................are in G.P.

   Then a_1k, a_2k, a_3k,....................are in G.P. & $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}$ ....are in G.P.

2. If a_1, a_2, a_3,................................are in G.P.

   Then $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ ....are in G.P.

3. If a_1, a_2, a_3,............................& b_1, b_2, b_3.............are in G.P.

   Then a_1b_1, a_2b_2, a_3b_3,..............are in G.P., & $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ ....are in G.P.

4. If a_1, a_2, a_3,.......................... & b_1, b_2, b_3,.............are in G.P having different common ratio then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3$,..............are not in G.P.

5. If a_1, a_2, a_3,....................... are in G.P.
Then $\log a_1, \log a_2, \log a_3$..........are in A.P. & Viceversa

6. If a_1, a_2, a_3,....................a_n are in G.P.
Then $a_1 a_n = a_2 a_{n-1} = a_3 . a_{n-2}$ = ...

7. If a_1, a_2, a_3,...........................are in G.P.
Then $a_2^2 = a_1 a_3 : a_3^2 = a_2 a_4 ; a_4^2 = a_3 a_5$;................

Selection of terms in an G.P.

No. of terms is odd	**Choosen terms**	**Common ratio**	**No. of terms is even**	**Choosen terms**	**Common ratio**
3	$\frac{a}{r}, a, ar$	r	4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5.	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r	6	$\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, a, ar^3, ar^5$	r^2

Practice Problems :

1. If a, b, c are in G.P. and $\log_c a$, $\log_b c$, $\log_a b$ are in A.P., then the common difference on the A.P. is

(a) 3 (b) 3/2 (c) 1/2 (d) 2/3

2. The value of $(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+....\text{to}\,\infty\right)}$ is

(a) 2 (b) 3 (c) 4 (d) none of these

3. If x, y, z are in GP and $a^x = b^y = c^z$, then

(a) $\log_b a = \log_a c$ (b) $\log_c b = \log_a c$ (c) $\log_b a = \log_c b$ (d) none of these

4. If a, b, c, d, e are in GP and $a^x = b^y = c^z = d^u = e^v$, then x, y, z, u, v are in

(a) AP (b) GP (c) HP (d) none of these

[Answers : (1) b (2) c (3) c (4) c]

C3 Arithmetic - Geometric Series :

A series is said to be an arithmetico-geometric series if its each term is formed by multiplying the corresponding term of an A.P. and a G.P.

e.g. $2x + 4x^2 + 6x^3 + 8x^4$ ....

Here 2, 4, 6, 8,..................are in A.P. and x, x^2, x^3, x^4..............are in G.P.

n^{th} Term of an Arithmetico-Geometric Series :

n^{th} term of arithmetico-geometric series can be obtained by multiplying the n^{th} term of a A.P & nth term of a G.P.

e.g. The n^{th} terms of the series $2x + 4x^2 + 6x^3 + 8x^4 + ...$ is

$\{2 + (n - 1)2\}(x . x^{n-1}) = 2n . x^n$

Sum of n terms of an arithmetico- geometric series :

Let $S_n = a + (a + d)r + (a + 2d)r^2 + ... + [a + (n - 1)d]r^{n-1}$ ....(i)

Multiplying both sides of (i) by common ratio of (G.P.) r and write as follows

$rS_n = ar + (a + d)r^2 + ... [a + (n - 1)d]r^n$ ....(ii)

From (i) – (ii)

$S_n(1-r) = a + [dr + dr^2 + \ldots\ldots + dr^{n-1}] - [a + (n-1)d]r^n$

$$\Rightarrow \quad S_n(1-r) = a + \frac{dr(1-r^{n-1})}{1-r} - [a+(n-1)d]r^n$$

$$\Rightarrow \quad S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{(1-r)}$$

Sum of Infinite Terms : $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

Practice Problems :

1. The sum if i – 2 – 3i + 4 ... upto 100 terms, where i = √–1 is

(a) 50(1 – i) (b) 25i (c) 25 (1 + i) (d) 100 (1 – i)

[Answers : (1) a]

C4 Harmonic Progression :

A sequence is said to be Harmonic Progression (H.P.) if the reciprocals of its terms are in Arithmetic Progression (A.P.)

If $a_1, a_2, a_3, \ldots\ldots$an are in H.P. then $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \ldots\ldots, \dfrac{1}{a_n}$ are in A.P.

1. nth term of this H.P. from start

$$T_n = \frac{1}{\frac{1}{a_1} + (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)} = \frac{a_1 a_2}{a_2 + (n-1)(a_1 - a_2)}$$

2. nth term of this H.P. from end

$$T_n' = \frac{1}{\frac{1}{a_n} - (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)} = \frac{a_1 a_2 a_n}{a_1 a_2 - a_n (n-1)(a_1 - a_2)}$$

3. $$\frac{1}{\text{nth term of H.P. from start}} + \frac{1}{\text{nth term of H.P. from last}}$$

i.e., $$\frac{1}{T_n} + \frac{1}{T_n'} = \frac{1}{a} + \frac{1}{a_n} = \frac{1}{\text{first term}} + \frac{1}{\text{last term}}$$

Practice Problems :

1. If b – c, 2b – x and b – a are in H.P., then $a - \left(\frac{x}{2}\right), b - \left(\frac{x}{2}\right) \text{and } c - \left(\frac{x}{2}\right)$ **are in :**

(a) A.P. (b) G.P. (c) H.P. (d) none of these

2. **If $\frac{a+b}{1-ab}, b, \frac{b+c}{1-bc}$ are in A.P., then a, $\frac{1}{b}$, c are in**

(a) **A.P.** **(b)** **G.P.** **(c)** **H.P.** **(d)** **none of these**

[Answers : (1) b (2) c]

C5A Arithmetic Mean, Geometric Mean and Harmonic Mean

	Arithmatic Mean	**Geometric Mean**	**Harmonic Mean**
Definition	If three terms in A.P. then the middle term is called the Arithmetic mean (A.M.) between the other two. e.g. If a, b, c are in A.P. then b is arithmetic mean of a and c	If three terms are in G.P. then middle term is called the Geometric mean (G.M.) between the other two. e.g. If a, b, c are in G.P., then b is geometric mean of a and c	If three terms are in H.P. then middle term is called the Harmonic mean (H.M.) between the other two. e.g. If a, b, c are in H.P., then b is harmonic mean of a and c
Single mean of n positive numbers $a_1, a_2......a_n$	$A = \frac{a_1 + a_2 + ... + a_n}{n}$ (A = Arithmetic mean)	$G = (a_1.a_2.a_3..........a_n)^{1/n}$ (G = Geometric mean)	$\frac{1}{H} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}$ (H = Harmonic mean)
	Special Case : If a and b are two given numbers then		
	$A = \frac{a+b}{2}$	$\mathbf{G = (ab)^{1/2}}$	$H = \frac{2ab}{a+b}$

C5B Properties of A.M., G.M. and H.M. :

1. If A, G and H be arithmetic and geometric means of two numbers a and b.
   Then, $\mathbf{G^2 = AH}$
2. The equation having a and b as its root is written in the form $x^2 - 2Ax + G^2 = 0$.
3. **Application to the questions of Inequalities :** In general A.M. ≥ G.M. ≥ H.M.

<u>Practice Problems :</u>

1. **The harmonic mean of two numbers is 4, their A.M. A, and G.M. G satisfy the relation $2A + G^2 = 27$. The two numbers are**

(a) **6, 3** **(b)** **5, 4** **(c)** **5, –2.5** **(d)** **–3, 1**

2. **If A_1, A_2 be two A.M.'s and G_1, G_2 be two G.M.'s between a and b, then $\frac{A_1 + A_2}{G_1 G_2}$ is equal to**

(a) $\frac{a+b}{2ab}$ **(b)** $\frac{2ab}{a+b}$ **(c)** $\frac{a+b}{ab}$ **(d)** $\frac{a+b}{\sqrt{ab}}$

[Answers : (1) a (2) c]

C6 Miscellaneous approach of summation :

1. $\sum$ Method :

Working Rule for Summation of Series :

- Find the n^{th} term of the series.
- Simplify the n^{th} terms.
- Now evaluate $S_n = \sum_{n=1}^{n} t_n$, with the help of $\sum n, \sum n^2, \sum n^3$.

$$\sum_{n=1}^{n} n = \frac{n(n+1)}{2}, \qquad \left(\sum_{n=1}^{n} n^2\right) = \frac{n(n+1)(2n+1)}{6} \qquad \text{and} \qquad \sum_{n=1}^{n} n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

2. **Method of differences :**

If the difference of the successive terms of a sequence is in A.P. or G.P., we find the n^{th} term of this sequence by method of difference, which is given as follows

$$S_n = t_1 + t_2 + t_3 + \ldots\ldots\ldots\ldots\ldots\ldots + t_n \qquad \ldots(1)$$

$$S_n = \qquad t_1 + t_2 + \ldots\ldots\ldots\ldots\ldots\ldots + t_{n-1} + t_n \qquad \ldots(2)$$

From (1) – (2); $0 = t_1 + (t_2 - t_1) + (t_3 - t_2) + \ldots.. + (t_n - t_{n-1}) - t_n$

$\Rightarrow \quad t_n = t_1 + (t_2 - t_1) + (t_3 - t_2) + \ldots. + (t_n - t_{n-1})$

So, $\quad S_n = \sum_{n=1}^{n} t_n$

<u>Practice Problems :</u>

1. The cubes of the natural numbers are grouped as 1^3, $(2^3, 3^3)$, $(4^3, 5^3, 6^3)$,... then the sum of the numbers in the nth group is

(a) $\frac{1}{8}n^3(n^2+1)(n^2+3)$ **(b)** $\frac{1}{16}n^3(n^2+16)(n^2+12)$

(c) $\frac{n^3}{12}(n^2+2)(n^2+4)$ **(d)** **none of the above**

2. The sum of n terms of the series $\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\cdots$ is

(a) $\frac{6n}{n+1}$ **(b)** $\frac{9n}{n+1}$ **(c)** $\frac{12n}{n+1}$ **(d)** $\frac{3n}{n+1}$

[Answers : (1) a (2) a]