Gravitation

C1 NEWTON'S LAW OF GRAVITATION

Every particle of matter in the universe attracts every other particle with a force, known as gravitational force. Newton's Law of Gravitation states that two particles with masses m_1 and m_2, a distance r apart, attract each other with gravitational forces of magnitude $\mathbf{F = \frac{Gm_1m_2}{r^2}}$.

Principle of Superposition :

If n particles interact, the net force $\vec{F}_{1,net}$ on a particle labeled as particle 1 is the sum of the forces on it from all the other particles taken one at a time.

$$\vec{F}_{1,net} = \sum_{i=2}^{n} \vec{F}_{1i}$$

In which the sum is a vector sum of the forces $\vec{F}_{1i}$ on particle 1 from particles 2, 3, ..., n. The gravitational force $\vec{F}_1$ on a particle from an extended body is found by dividing the body into units of differential mass dm, each of which produces a differential force $d\vec{F}$ on the particle, and then integrating to find the sum of those forces :

$$\vec{F}_1 = \int d\vec{F}$$

Gravitation Behaviour of spherically symmetric solid :

The gravitational effect outside any spherical symmetric mass distribution is the same as though all the mass of the sphere were concentrated at its center.

Practice Problems :

1. **A point mass of mass M are broken into two parts and separated by certain distance d. The maximum force between the two parts is given by**

(a) $\frac{GM^2}{2d^2}$ (b) $\frac{GM^2}{4d^2}$ (c) $\frac{GM^2}{3d^2}$ (d) $\frac{GM^2}{8d^2}$

2. **Three point masses each of mass m are placed at the corner of an equilateral triangle of side length d. The force on one of the mass is**

(a) $\frac{GM^2}{d^2}$ (b) $\frac{GM^2}{3d^2}$ (c) $\frac{\sqrt{3}GM^2}{d^2}$ (d) $\frac{2\sqrt{3}GM^2}{d^2}$

[Answers : (1) b (2) c]

C2 GRAVITATION FIELD AND INTENSITY

The magnitude gravitational intensity due to a mass m at point P is $\mathbf{E = \frac{Gm}{r^2}}$, directed toward the mass m.

Principle of Superposition :

In the presence of n particles, the net gravitational intensity $\vec{E}$ at a particular point is the vector sum of the intensities due to individual particles at that particular point : $\vec{E} = \sum_{i=1}^{n} \vec{E}_i$.

The gravitational intensity due to an extended body is found by dividing the body into units of differential mass dm, each of which produces a differential intensity $d\vec{E}$, and then integrate to find the gravitational intensity : $\vec{E} = \int d\vec{E}$.

Practice Problems :

1. **Two uniform solid spheres of equal radii R, but mass M and 4 M have a centre to centre separation 6 R. The two spheres are held fixed. Find the distance from M at which the gravitational field intensity will be zero ?**
2. **What is the gravitational field intensity due to uniform ring at the centre ?**
3. **Find the gravitational field intensity due to a uniform ring of mass M and radius R at the distance x from the centre of the ring on its axis ?**

[Answers : (1) 2R (2) zero (3) $\frac{GMx}{(x^2+R^2)^{3/2}}$]

C3 GRAVITATION FIELD INTENSITY DUE TO EARTH AND ACCELERATION DUE TO GRAVITY

The gravitational field intensity due to earth and acceleration due to gravity are the same.

Outside or on the Earth : The value of g outside the earth at the distance r from the center is $g = \frac{GM}{r^2}$.

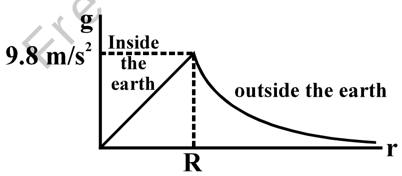
On the surface of earth, $g_0 = \frac{GM}{R^2}$ where R is the radius of earth.

Above the surface of height h, g is given by, $g = g_0 \frac{R^2}{(R+h)^2} = g_0\left(1+\frac{h}{R}\right)^{-2}$

Using the approximation $(1 + x)^n \approx 1 + nx$ for $|x| << 1$, we get $g \approx g_0\left(1-\frac{2h}{R}\right)$ if $h << R$.

Inside the Earth : The value of g inside the earth at the distance r is given by $g = g_0\left(\frac{r}{R}\right)$

Variation of g with positon :



Effect of 'g' due to Earth's rotation : $g = g_0 - \omega^2 R\cos^2\theta$, where θ is the angle of latitude and ω is the angular velocity of the earth about its axis.

case I At the equator, $\theta = 0$, $g = g_0 - \omega^2 R$

case II At the pole, $\theta = \frac{\pi}{2}$, $g = g_0$.

Hence there is no effect of earth's rotation on g at pole.

Practice Problems :

1. **If the radius of the earth were to shrink by one per cent, its mass remaining the same, the value of g on the earth's surface would**

(a)	**increase by 0.5%**	**(b)**	**increase by 2 %**
(c)	**decrease by 0.5%**	**(d)**	**decrease by 2 %**

2. **Two planets have radii R_1 and R_2 and densities ρ_1 and ρ_2 respectively. The ratio of the acceleration due to gravity at their surface is**

(a) $\frac{\rho_1 R_1}{\rho_2 R_2}$ (b) $\frac{\rho_1 R_2^2}{\rho_2 R_1^2}$ (c) $\frac{R_1 R_2}{\rho_1 \rho_2}$ (d) $\frac{\rho_2 R_1}{\rho_1 R_2}$

3. **If the value of g at the surface of the earth is 9.8 m/s^2, then the value of g at a place 480 km above the surface of the earth will be (radius of earth = 6400 km)**

(a) **4.2 m/s^2** (b) **7.2 m/s^2** (c) **8.5 m/s^2** (d) **9.8 m/s^2**

4. **If R is the radius of the earth then the altitude at which the acceleration due to gravity will be 25% of its value at the earth's surface is**

(a) **R/4** (b) **R** (c) **3R/8** (d) **R/2**

5. **The radius of the earth is 6400 km and g = 10 m/s^2. In order that a body of 5 kg weight zero at the equator, the angular speed of the earth should be**

(a) $\frac{1}{80}$ rad/s (b) $\frac{1}{400}$ rad/s (c) $\frac{1}{800}$ rad/s (d) $\frac{1}{1600}$ rad/s

6. **As we go from the equator to the poles, the value of g**

(a) **remains the same** (b) **increases**

(c) **decreases** (d) **decrease up to a latitude of 45^0.**

[Answers : (1) b (2) a (3) c (4) b (5) c (6) b]

C4 GRAVITATIONAL POTENTIAL ENERGY :

The gravitational potential energy U(r) of a system of two particles, with masses M and m and separated by a distance r, is given by $\mathbf{U = -\frac{GMm}{r}}$.

Potential energy of the system of particles :

If a system contains more than two particles, its total gravitational potential energy U is the sum of terms representing the potential energies of all the pairs. As an example, for three particles, of masses m_1, m_2 and m_3

$$\mathbf{U = -\left(\frac{G_1 m_1 m_2}{r_{12}} + \frac{G m_1 m_3}{r_{13}} + \frac{G m_2 m_3}{r_{23}}\right)}$$ where r_{12} is ....

Relation between potential energy and force : $\mathbf{\vec{F} = -\frac{dU}{dr}\hat{r}}$

Practice Problems :

1. **A body of mass m is taken from the earth's surface to a height equal to the radius R of the earth. If g is the acceleration due to gravity at the surface of the earth, then the change in the potential energy of the body is**

(a) $\mathbf{\frac{1}{4}mgR}$ (b) $\mathbf{\frac{1}{2}mgR}$ (c) **mgR** (d) **2 mgR**

2. **Three point masses each of mass 'm' are placed on an equilateral triangle of side length *l*. Find the potential energy of this system of particles ?**

[Answers : (1) b (2) $\frac{-3Gm^2}{l}$]

C5 GRAVITATION POTENTIAL (V) :

It is defined as the Gravitational Potential Energy per unit mass. The Gravitational Potential (V) due to mass m at point P is $-\frac{Gm}{r}$.

Relation between Gravitational Potential (V) and intensity $(\vec{E}) : \vec{E} = -\frac{dV}{dr}\hat{r}$

Practice Problems :

1. **The gravitational field due to a mass distribution is E = K/x³ in the x-direction, where K is a constant. Taking the gravitational potential to be zero at infinity, its value at a distance x is**

   (a) K/x (b) K/2x (c) K/x^2 (d) $K/2x^2$

[Answers : (1) d]

C6 ESCAPE SPEED

An object will escape from the gravitational pull of an astronomical body if the object is projected with a certain minimum speed from the body is surface and this minimum speed is known as escape speed. Escape speed from the surface of the earth is $V_e = \sqrt{\frac{2GM}{R}}$, where M is the mass of the earth and R is the radius.

Let the astronomical body is of uniform density ρ then $M = \frac{4}{3}\pi R^3 \rho$ and hence $V_e = \sqrt{\frac{8}{3}\pi G R^2 \rho}$. The escape speed of an object at a given point in the field is independent of its mass and state of its motion, but is position-dependent. For earth the escape speed from the surface V_e is 11.2 km/s.

Practice Problems :

1. **The escape velocity from the earth is 11 km/s. The escape velocity from a planet having twice the radius and the same mean density as those of the earth is**

   (a) 5.5 km/5 (b) 11 km/s (c) 22 km/s (d) none of these

2. **The mass of moon is $\frac{1}{81}$ of earth's mass and its radius is $\frac{1}{4}$ of that of the earth. If the escape velocity from the earth's surface is 11.2 km/s, its value for the moon is**

   (a) 0.14 km/s (b) 0.5 km/s (c) 2.5 km/s (d) 5.0 km/s

3. **The escape velocity of a particle of mass m varies as Km^{α}, where K is a constant. The value of α is**

   (a) 0 (b) 1 (c) ½ (d) –½

4. **If a rocket is to be projected vertically upwards from the surface of the earth, it requires an escape velocity of 11 km/s. If the rocket is to be projected at an angle of 60^0 with the vertical, the escape velocity required will be about**

   (a) 5.5 km/s (b) 11 √2 km/s (c) 11 km/s (d) 5.5 × √3 km/s

[Answers : (1) c (2) c (3) a (4) c]

C7 SATELLITES : ORBITS AND ENERGY

A satellite is any body revolving around a larger body, under the influence of the latter. For example the moon is a gravitational satellite of the earth. There are various artificial satellite of the earth, they are known as geo-static or geo stationary or geo-synchronous satellite. For this type of satellites, the conditions are

(a) the orbit must be circular

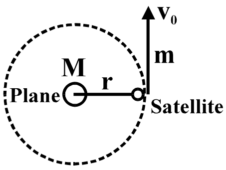
(b) the orbit must be in equatorial plane of the earth

(c) the period of revolution of the satellites is 24 hour

(d) the angular velocity of revolution of the satellite must be in the same direction as the angular velocity of rotation of the earth. If a satellite is to be always seen overhead, the observer should be on the equator of the Earth.

For a satellite in circular orbit :

Orbital speed $\mathbf{v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R+h}}}$



where r = R + h, R is the radius of planer and h is height from the surface of planet at which a satellite is revolving.

Time Period $\mathbf{T = \frac{2\pi r}{v_0} = \sqrt{\frac{4\pi^2}{GM} r^3}}$, Kinetic energy $\mathbf{K = \frac{1}{2} m v_0^2 = \frac{GMm}{2r}}$,

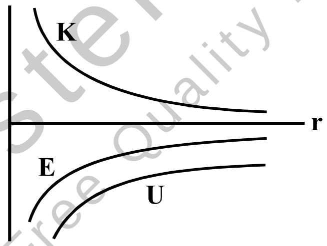
Potential energy $\mathbf{U = -\frac{GMm}{r}}$, Total mechanical energy = E = K + U = $-\frac{\mathbf{GMm}}{\mathbf{2r}}$

The negative sign indicates that the satellite is bound.

Relation between E, U and K : $\mathbf{E = -K, E = \frac{U}{2}, K = -\frac{U}{2}}$

Binding energy of the satellite = $\mathbf{\frac{GMm}{2r}}$

Variation of K, U and E with r is shown in figure.



Speeds and nature of orbits :

(a) If $V < V_0$; the orbit is elliptical with the centre of the earth as the further focus, E is negative.

(b) If $V = V_0$; the orbit is circular, E is negative

(c) If $V_0 < V < V_e$: the orbit is elliptical, E is negative

(d) If $V = V_e$: the body escapes, E is zero.

(e) If $V > V_e$: the body escapes along a hyperbolic Path, E is positive.

Here E stands for total mechanical energy of the body.

<u>Practice Problems :</u>

1. A satellite of mass m is revolving around the earth at a height R above the surface of the earth. If g is the gravitational field intensity at the earth's surface and R is the radius of the earth, the kinetic energy of the satellite is

(a) $\frac{mgR}{4}$ **(b)** $\frac{mgR}{2}$ **(c)** **mgR** **(d)** **2 mgR**

2. The orbital velocity of an artificial satellite in a circular orbit just above the earth's surface is v. For a satellite orbiting at an altitude of half the earth's radius, the orbital velocity is

(a) $\frac{3}{2}$ v (b) $\sqrt{\frac{3}{2}}$v (c) $\sqrt{\frac{2}{3}}$v (d) $\frac{2}{3}$ v

3. Let the total energy, kinetic energy and potential energy of the artificial satellite are E_1, E_2 and E_3 respectively. Then $E_1 : E_2 : E_3$ is

(a) 1 : 1 : 2 (b) –1 : 1 : –2 (c) –1 : 1 : 2 (d) 1 : 1 : –2

4. Time period of revolution of a satellite close to the surface of a spherical planet of radius R is T. The period of revolution close to the surface of another planet of radius 3R and same density is

(a) T (b) 3T (c) $3\sqrt{3}$ T (d) 9T

[Answers : (1) a (2) c (3) b (4) a]

C8 KEPLER'S LAW

(i) First Law (law of orbit)

The planets move around the sun in elliptical orbits with the sum at one focus.

(ii) Second Law (law of area)

The line joining the sun to a planet sweeps out equal areas in equal times. This law is based on conservation of angular momentum.

(iii) Third Law (law of period)

The square of the period of planet is proportional to the cube of its mean distance from the sun.

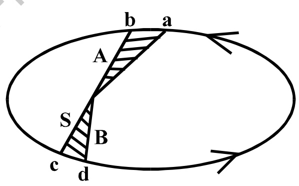
The mean distance turns out to be the semi-major axis, a, i.e., $T^2 \propto a^3$

Practice Problems :

1. A planet moves around the sun. At a point P it is closed to the sun at a distance d_1 and has a speed v_1. At another point Q, when it is farthest from the sun at a distance d_2, its speed will be

(a) $\frac{d_1^2 v_1}{d_2^2}$ (b) $\frac{d_2 v_1}{d_1}$ (c) $\frac{d_1 v_1}{d_2}$ (d) $\frac{d_2^2 v_1}{d_1^2}$

2. The figure shows the motion of a planet around the sun in an elliptic orbit with the sun at one focus. The shaded areas A and B can be assumed to be equal. If t_1 and t_2 represent the times taken by the planet to move from a to b and from c to d respectively, then



(a) $t_1 < t_2$

(b) $t_1 > t_2$

(c) $t_1 = t_2$

(d) from the given information the relation between t_1 and t_2 cannot be determined.

3. Kepler's second law is based on

(a) conservation of energy (b) conservation of linear momentum

(c) conservation of angular momentum (d) conservation of mass

[Answers : (1) c (2) c (3) c]