TRIGONOMETRIC FUNCTIONS AND TRIGONOMETRIC EQUATIONS

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C1 Trigonometric Functions :

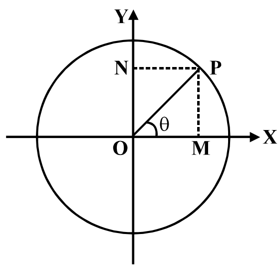
Basic Trigonometric Identities :

(a) $\sin^2\theta + \cos^2\theta = 1; -1 \le \sin\theta \le 1; -1 \le \cos\theta \le 1 \;\forall\; \theta \in R$

(b) $\sec^2\theta - \tan^2\theta = 1; |\sec\theta| \ge 1 \;\forall\; \theta \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

(c) $\text{cosec}^2\theta - \cot^2\theta = 1 ; |\text{cosec}\,\theta| \ge 1 \;\forall\; \theta \in R - \{n\pi, n \in I\}$

Circular Definition of Trigonometric Functions :

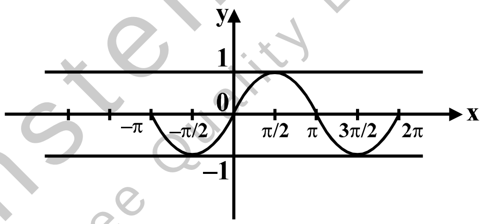


$\sin\theta = \frac{PM}{OP}, \cos\theta = \frac{OM}{OP}$, $\tan\theta = \frac{\sin\theta}{\cos\theta}, \cos\theta \ne 0$, $\cot\theta = \frac{\cos\theta}{\sin\theta}, \sin\theta \ne 0$, $\sec\theta = \frac{1}{\cos\theta}, \cos\theta \ne 0$

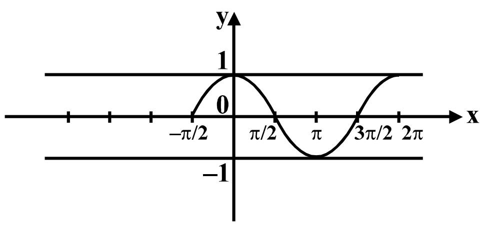
$\text{cosec}\,\theta = \frac{1}{\sin\theta}, \sin\theta \ne 0$

C2 Graphs of Trigonometric functions :

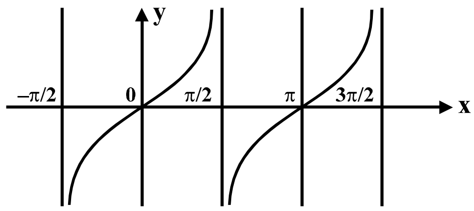
(a) $y = \sin x,\; x \in R; y \in [-1, 1]$



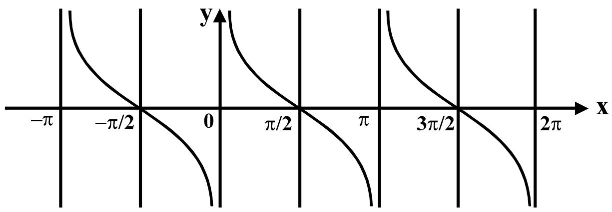
(b) $y = \cos x,\; x \in R; y \in [-1, 1]$



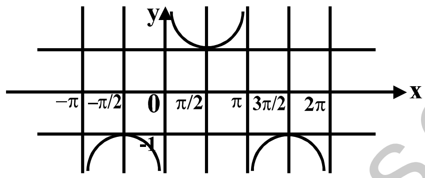
(c) $y = \tan x,\; x \in R - (2n + 1)\,\pi/2, n \in I; y \in R$



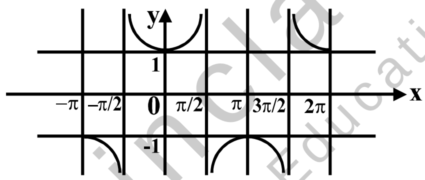
(d) $y = \cot x,\ x \in R - n\pi, n \in I; y \in R$



(e) $y = \text{cosec}\, x,\ x \in R - n\pi, n \in I; y \in (-\infty, -1] \cup [1, \infty)$



(f) $y = \sec x,\ x \in R - (2n + 1)\pi/2, n \in I, y \in (-\infty, -1] \cup [1, \infty)$



Practice Problems :

1. The value of $\cos^2 5^0 + \cos^2 10^0 + \ldots\ldots \cos^2 85^0 + \cos^2 90^0$ will be

(a) $\frac{17}{2}$ **(b)** $\frac{19}{2}$ **(c)** 1 **(d)** 0

2. If $\sec\theta - \tan\theta = k$ then the value of $\cos\theta$ will be

(a) $\frac{2k}{k^2+1}$ **(b)** $\frac{2k}{k^2-1}$ **(c)** $\frac{k^2+1}{2k}$ **(d)** $\frac{k^2-1}{2k}$

3. If cosec A + cot A = $\frac{11}{2}$ then tan A equals

(a) $\frac{21}{22}$ **(b)** $\frac{15}{16}$ **(c)** $\frac{44}{117}$ **(d)** $\frac{117}{43}$

[Answers : (1) a (2) a (3) c]

C3 Trigonometric Functions of Allied Angles :

If θ is any angle, then –θ, 90 ± θ, 180 ± θ, 270 ± θ, 360 ± θ etc. are called Allied Angles.

(a) $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$

(b) $\sin(90^0 - \theta) = \cos\theta$; $\cos(90^0 - \theta) = \sin\theta$

(c) $\sin(90^0 + \theta) = \cos\theta$; $\cos(90^0 + \theta) = -\sin\theta$

(d) $\sin(180^0 - \theta) = \sin\theta$; $\cos(180^0 - \theta) = -\cos\theta$

(e) $\sin(180^0 + \theta) = -\sin\theta$; $\cos(180^0 + \theta) = -\cos\theta$

(f) $\sin(270^0 - \theta) = -\cos\theta$; $\cos(270^0 - \theta) = -\sin\theta$

(g) $\sin(270^0 + \theta) = -\cos\theta$; $\cos(270^0 + \theta) = \sin\theta$

(h) $\tan(90^0 - \theta) = \cot\theta$; $\cot(90^0 - \theta) = \tan\theta$

<u>Practice Problems :</u>

1. If $\sin\theta = \frac{1}{2}$ and θ is obtuse then cot θ equals

(a) $\frac{1}{\sqrt{3}}$ **(b)** $-\frac{1}{\sqrt{3}}$ **(c)** $\sqrt{3}$ **(d)** $-\sqrt{3}$

2. The value of sin 10^0 + sin 20^0 + sin 30^0......sin 360^0 is

(a) 0 **(b)** 1 **(c)** –1 **(d)** none of these

3. The value of cos 1^0 . cos 2^0 . cos 3^0.......cos 180^0 will be

(a) 0 **(b)** 1

(c) 100 **(d)** cannot be found

4. The value of cos 1^0 + cos 2^0 + cos 3^0.......cos 180^0 will be

(a) 0 **(b)** –1 **(c)** 1 **(d)** none of these

5. The value of log [cos 1^0 + cos 2^0 + cos 3^0 ..... cos 180^0] will be

(a) 0 **(b)** 1 **(c)** –1 **(d)** not defined

6. The value of log [cot 1^0 . cot 2^0 . cot 3^0 ..... cot 89^0] will be

(a) 0 **(b)** 1 **(c)** –1 **(d)** not defined

[Answers : (1) d (2) a (3) a (4) b (5) d (6) a]

C4 Trigonometric Functions of Sum and Difference of Two Angles :

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A + B) . \sin(A - B)$

(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A + B) . \cos(A - B)$

(e) $$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

(f) $$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$$

(g) $$\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

Practice Problems :

1. If sin A + cos B = a and sin B + cos A = b then the value of sin (A + B) will be

(a) $a^2 + b^2$ (b) $\frac{a^2+b^2}{2}$ (c) $\frac{a^2+b^2-2}{2}$ (d) $\frac{a^2+b^2+2}{2}$

2. If $\tan(\alpha+\beta) = \frac{1}{2}$ and $\tan(\alpha-\beta) = \frac{1}{3}$ then $\tan 2\alpha$ will be

(a) 0 (b) 1 (c) –1 (d) none of these

3. If $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$ then $\alpha + \beta$ will be

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

4. The value of $\cos^2 48^0 - \sin^2 12^0$ equals to

(a) $\frac{\sqrt{5}-1}{4}$ (b) $\frac{\sqrt{5}+1}{4}$ (c) $\frac{\sqrt{5}+1}{8}$ (d) $\frac{\sqrt{5}-1}{8}$

5. The value of $\sqrt{3}$ cosec 20^0 – sec 20^0 is equal to

(a) 2 (b) $\frac{2\sin 20^0}{\sin 40^0}$ (c) 4 (d) $\frac{4\sin 20^0}{\sin 40^0}$

6. The value of tan 5x . tan 3x . tan 2x is equal to

(a) tan 5x – tan 3x – tan 2x (b) $\frac{\sin 5x - \sin 3x - \sin 2x}{\cos 5x - \cos 3x - \cos 2x}$

(c) 0 (d) none of these

7. The value of $\frac{\cos 9^0 + \sin 9^0}{\cos 9^0 - \sin 9^0}$ equal to

(a) $\tan 54^0$ (b) $\tan 36^0$ (c) $\tan 18^0$ (d) none of these

8. If $A + B + C = \pi$ (A, B, C > 0) and the angle C is obtuse then

(a) tan A . tan B > 1 (b) tan A . tan B < 1

(c) tan A . tan B = 1 (d) none of these

[Answers : (1) c (2) b (3) b (4) c (5) c (6) a (7) a (8) b]

C5 Factorisation of the Sum and Difference of Two Sines and Cosines :

(a) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(b) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(d) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

C6 Transformation of Products into Sum and Difference of Sines & Cosines :

(a) $2\sin A\cos B = \sin(A + B) + \sin(A - B)$

(b) $2\cos A\sin B = \sin(A + B) - \sin(A - B)$

(c) $2\cos A\cos B = \cos(A + B) + \cos(A - B)$

(d) $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

C7 Multiple and Sub-multiple Angles :

(a) $\sin 2A = 2\sin A\cos A$; $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$; $2\cos^2\frac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\frac{\theta}{2} = 1 - \cos\theta$.

(c) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}; \tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(d) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}; \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(e) $\sin 3A = 3\sin A - 4\sin^3 A$

(f) $\cos 3A = 4\cos^3 A - 3\cos A$

(g) $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

C8 Important Trigonometric Ratios :

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$, where $n \in I$

(b) $\sin 15^0 \text{ or } \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^0 \text{ or } \cos\frac{5\pi}{12}$

$\cos 15^0 \text{ or } \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^0 \text{ or } \sin\frac{5\pi}{12}$

$\tan 15^0 = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75^0; \tan 75^0 = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15^0$

(c) $\sin\frac{\pi}{10} \text{ or } \sin 18^0 = \frac{\sqrt{5}-1}{4}$ & $\cos 36^0 \text{ or } \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

Practice Problems :

1. If sin A + sin B = a and cos A + cos B = b then the value of sin (A + B) will be

(a) $\frac{2ab}{a^2+b^2}$ (b) $\frac{2ab}{a^2-b^2}$ (c) $\frac{a^2+b^2}{2ab}$ (d) $\frac{a^2-b^2}{2ab}$

2. The value of $\frac{\sin 70^0 + \cos 40^0}{\cos 70^0 + \sin 40^0}$ equal to

(a) 1 (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $\frac{1}{2}$

3. If $\frac{\sin A - \sin C}{\cos C - \cos A} = \cot B$, then A, B, C are in

(a) A.P. (b) G.P. (c) H.P. (d) none of these

4. The value of $\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n$ equals

(a) $2\tan^n \frac{A-B}{2}$ (b) $2\cot^n \frac{A-B}{2}$

(c) 0 (d) both (b) and (c) are correct

5. The value of $\sin 12^0 . \sin 24^0 . \sin 48^0 . \sin 84^0$ will be

(a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $\frac{1}{32}$ (d) $\frac{1}{64}$

6. The value of $\sin 36^0 . \sin 72^0 . \sin 108^0 . \sin 144^0$ will be

(a) $\frac{1}{4}$ (b) $\frac{1}{16}$ (c) $\frac{3}{4}$ (d) $\frac{5}{16}$

7. The value of $\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8}$

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{2}$ (d) $\frac{3}{4}$

8. The value of

$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ will be

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

9. The value of $\sin\left(\frac{\pi}{10}\right)\sin\left(\frac{3\pi}{10}\right)$ equals to

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{4}$ (d) 1

10. The value of $\cos\frac{\pi}{5}.\cos\frac{2\pi}{5}.\cos\frac{4\pi}{5}.\cos\frac{8\pi}{5}$ will be

(a) $\frac{1}{16}$ (b) 0 (c) $-\frac{1}{8}$ (d) $-\frac{1}{16}$

11. The value of $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$ equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

12. Prove the following statements :

(a) $\cos^4 A - \sin^4 A + 1 = 2\cos^2 A$ (b) $(\sin A + \cos A)(1 - \sin A . \cos A) = \sin^3 A + \cos^3 A$

(c) $\frac{\sin A}{1+\cos A} + \frac{1+\cos A}{\sin A} = 2\,\mathrm{cosec}A$ (d) $\cos^6 A + \sin^6 A = 1 - 3\sin^2 A . \cos^2 A$

(e) $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A$ (f) $\dfrac{\operatorname{cosec} A}{\cot A + \tan A} = \cos A$

(g) (1 + cotA – cosecA) (1 + tanA + secA) = 2

(h) $\dfrac{1}{\operatorname{cosec} A - \cot A} - \dfrac{1}{\sin A} = \dfrac{1}{\sin A} - \dfrac{1}{\operatorname{cosec} A + \cot A}$

(i) $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A . \tan B$

(j) $\left(\dfrac{1}{\sec^2\alpha - \cos^2\alpha} + \dfrac{1}{\operatorname{cosec}^2\alpha - \sin^2\alpha}\right)\cos^2\alpha.\sin^2\alpha = \dfrac{1-\cos^2\alpha.\sin^2\alpha}{2+\cos^2\alpha.\sin^2\alpha}$

[Answers : (1) a (2) c (3) a (4) d (5) b (6) d (7) c (8) c (9) c (10) d (11) c]

C9 Range of Trigonometric Expression :

E = a sin θ + b cos θ

$E = \sqrt{a^2+b^2}$ sin (θ + α), where $\tan\alpha = \dfrac{b}{a}$

$= \sqrt{a^2+b^2}\cos(\theta-\beta)$, where $\tan\beta = \dfrac{a}{b}$

Hence for any real value of $\theta, -\sqrt{a^2+b^2} \le E \le \sqrt{a^2+b^2}$

Practice Problems :

1. **Find the value of x for which following expression will have maximum value :**

   (a) $\sqrt{3}$ cos x + sin x (b) cos x + sin x

2. **Find the range of following trigonometric expression**

   (a) $\sqrt{3}$ cos x + sin x (b) cos x + sin x

   (c) 3sin x + 4 cos x

C10 Sine and Cosine Series :

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ... + \sin\left(\alpha + \overline{n-1}\,\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ... + \cos\left(\alpha + \overline{n-1}\,\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

C11 Trigonometric Equations : Equation involving trigonometric functions of a variable are called trigonometric equations. The trigonometric equation may have infinite number of a solutions and can be classified as :

(i) Principal solution : The solution of a trigonometric equations for which 0 ≤ x < 2π are called principal solution.

(ii) General solution

Important Points :

1. $\sin\theta = 0 \Leftrightarrow \theta = n\pi$
2. $\cos\theta = 0 \Leftrightarrow \theta = (2n+1)\frac{\pi}{2}$
3. $\tan\theta = 0 \Leftrightarrow \theta = n\pi$
4. $\sin\theta = \sin\alpha \Leftrightarrow \theta = n\pi + (-1)^n\alpha$, where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
5. $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$
6. $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$
7. $\sin^2\theta = \sin^2\alpha$, $\cos^2\theta = \cos^2\alpha$, $\tan^2\theta = \tan^2\alpha \Leftrightarrow \theta = n\pi \pm \alpha$
8. $\sin\theta = 1 \Leftrightarrow \theta = (4n+1)\frac{\pi}{2}$
9. $\cos\theta = 1 \Leftrightarrow \theta = 2n\pi$
10. $\cos\theta = -1 \Leftrightarrow \theta = (2n+1)\pi$
11. $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha \Leftrightarrow \theta = 2n\pi + \alpha$

Practice Problems :

1. The most general value of θ satisfying the equation $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$ is

(a) $n\pi + \frac{7\pi}{4}$ (b) $n\pi + (-1)^n\frac{7\pi}{4}$ (c) $2n\pi + \frac{7\pi}{4}$ (d) none of these

2. The number of solution of the given equation $\tan\theta + \sec\theta = \sqrt{3}$ where $0 < \theta < 2\pi$ is

(a) 0 (b) 1 (c) 2 (d) 3

3. If $3(\sec^2\theta + \tan^2\theta) = 5$ then the general solution of θ is

(a) $2n\pi \pm \frac{\pi}{6}$ (b) $n\pi \pm \frac{\pi}{6}$ (c) $2n\pi \pm \frac{\pi}{3}$ (d) $n\pi \pm \frac{\pi}{3}$

4. If $\sin 3\theta = \sin\theta$ then the general value of θ is

(a) $2n\pi, (2n+1)\frac{\pi}{3}$ (b) $n\pi, (2n+1)\frac{\pi}{4}$ (c) $n\pi, (2n+1)\frac{\pi}{3}$ (d) none of these

5. If $\tan m\theta = \tan n\theta$ then then the consecutive value of θ will be in

(a) A.P. (b) G.P. (c) H.P. (d) none of these

6. The number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is

(a) 0 (b) 1 (c) 2 (d) 3

7. The complete solution of the equation $7\cos^2x + \sin x\cos x - 3 = 0$ is given by

(a) $n\pi + \frac{\pi}{2} (n \in I)$ (b) $n\pi - \frac{\pi}{4} (n \in I)$

(c) $n\pi + \tan^{-1}\left(\frac{4}{3}\right) (n \in I)$ (d) $n\pi + \frac{3\pi}{4}, k\pi + \tan^{-1}\left(\frac{4}{3}\right)$ (k, n ∈ I)

8. If $\sin\theta + \cos\theta = \sqrt{2}\cos\alpha$ then the general value of θ is

(a) $2n\pi - \frac{\pi}{4} \pm \alpha$ (b) $2n\pi + \frac{\pi}{4} \pm \alpha$ (c) $n\pi - \frac{\pi}{4} \pm \alpha$ (d) $n\pi + \frac{\pi}{4} \pm \alpha$

9. The general solution of the equation $\sin^{100}x - \cos^{100}x = 1$ is

(a) $2n\pi + \frac{\pi}{3}, n \in I$ (b) $2n\pi + \frac{\pi}{2}, n \in I$

(c) $n\pi + \frac{\pi}{4}, n \in I$ (d) $2n\pi - \frac{\pi}{3}, n \in I$

[Answers : (1) c (2) c (3) b (4) b (5) a (6) d (7) d (8) b (9) b]

ADDITIONAL PRACTICE PROBLEMS

1. **Prove that**

$$\sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) = 0$$

2. **If A + B + C = π, then prove that**

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

(iii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(iv) $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi)

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

(vii)

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}$$

(viii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

3. **If $A + B + C = \frac{\pi}{2}$ then prove that $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$**

4. **If $A + B = 45^0$ show that $(1 + \tan A)(1 + \tan B) = 2$**

5. **Prove that :**

(a) $$\frac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \tan\theta$$

(b) $$\frac{\sin 7A - \sin A}{\sin 8A - \sin 2A} = \cos 4A.\sec 5A$$

(c) $$\frac{\cos 3A - \cos A}{\sin 3A - \sin A} + \frac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} = \frac{\sin A}{\cos 2A.\cos 3A}$$

(d) $$\frac{\sin(4A - 2B) + \sin(4B - 2A)}{\cos(4A - 2B) + \cos(4B - 2A)} = \tan(A + B)$$

(e) $$\frac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$$

(f) $$\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}$$