TANGENT AND NORMAL

C1. Derivative as rate of change

If the quantity y varies with respect to another quantity x satisfying some relation y = f(x), then $f'(x)$ or $\frac{dy}{dx}$ represents rate of change of y with respect to x.

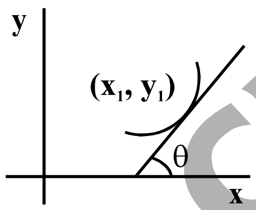
Practice Problems :

1. **The surface area of a balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 2 seconds, it is 5 units, find the radius after t seconds.**

2. **On the curve $x^3 = 12y$, find the interval at which the abscissa changes at a faster rate than the ordinate ?**

[Answers : (1) $r = \sqrt{8t+9}$ (2) $x \in (-2, 2) - \{0\}$]

C2. Equation of Tangent and Normal



$\tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = f'(x_1)$ denotes the slope of tangent at point (x_1, y_1) on the curve y = f(x) as shown in figure. Hence the equation of tangent at (x_1, y_1) is given by

$$(y - y_1) = f'(x_1)(x - x_1)$$

Also, since normal is a line perpendicular to tangent at (x_1, y_1) so its equation is given by

$$(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$$

Practice Problems :

1. **Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.**

2. **If the tangent to the curve $y = x^3 + ax + b$ at (1, –6) is parallel to the line x – y + 5 = 0, find the values of a and b.**

3. **Find the equation of the tangent line to the curve**

   $x = \theta + \sin\theta$, $y = 1 + \cos\theta$ at $\theta = \pi/4$.

4. **Find the point on the curve $y = x^3 - 3x$ where the tangent is parallel to the chord joining (1, –2) and (2, 2).**

5. **Find the equation of the tangent to the curve $\sqrt{x} + \sqrt{y} = a$ at the point $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$.**

6. **If the line ax + by + c = 0 is normal to the curve xy + 5 = 0, then show a and b have same sign.**

7. **If the tangent at (x_0, y_0) to the curve $x^3 + y^3 = a^3$ meets the curve again at (x_1, y_1) then prove that $\frac{x_1}{x_0} + \frac{y_1}{y_0} = 1$?**

8. **Find the equation of tangent and normal to the curve $y^2(a + x) = x^2(3a - x)$ at the point where $x = a$.**

9. **Find the point on the curve $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 2$, so that it touches the line $\frac{x}{a} + \frac{y}{b} = 2$.**

[Answers : (1) 1 (2) a = –2, b = –5 (3) $y - \left(1 + \frac{1}{\sqrt{2}}\right) = (1 - \sqrt{2})\left[x - \left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)\right]$ (4) $x = \pm\sqrt{\frac{7}{3}}, y = \mp\frac{2}{3}\sqrt{\frac{7}{3}}$ (5) $x + y = a^2/2$ (8) $x + 2y + a = 0$, $2x - y - 3a = 0$ (9) (a, b)]

C3. Length of Tangent and Normal

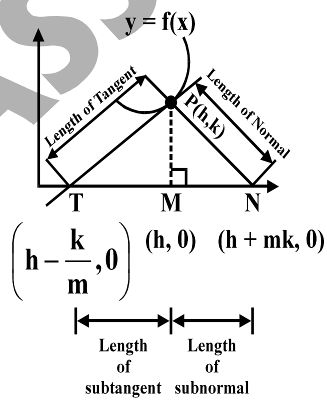
Let P (h, k) be any point on curve y = f(x). Let tangent drawn at point P meets x-axis at T & normal at point P meets x-axis at N.

PT = Length of tangent

PN = Length of normal

TM = Length of subtangent

MN = Length of subnormal



Let $m = \left.\frac{dy}{dx}\right|_{h,k}$ = slope of tangent.

Hence equation of tangent is m(x – h) = (y – k)

putting y = 0 we get x-intercept of tangent $x = h - \frac{k}{m}$

similarly the x-intercept of normal is x = h + km

Now, length PT, PN etc can be easily evaluated using distance formula

(i) $PT = \left|k\sqrt{1 + \frac{1}{m^2}}\right|$ = Length of Tangent

(ii) $PN = \left|k\sqrt{1 + m^2}\right|$ = Length of Normal

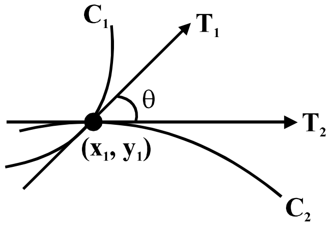
(iii) $TM = \left|\frac{k}{m}\right|$ = Length of subtangent

(iv) **MN** = |km| = Length of subnormal

C4. Angle between the curves

Angle between two intersecting curves is defined as the acute angle between their tangents or the normals at the point of intersection of two curves.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$



where m_1 & m_2 are the slopes of tangents at the intersection point (x_1, y_1). Note carefully that

(i) The curves must intersect for the angle between them to be defined. This can be ensured by finding their point of intersection of graphically.

(ii) If the curves intersect at more than one point then angle between curves is written with reference to the point of intersection.

(iii) Two curves are said to be orthogonal if angle between them at each point of intersection is right angle. i.e. $m_1 m_2 = -1$.

(iv) If the tangents of two curves are paralle to each other then $m_1 = m_2$.

(v) If any tangent of curve is equally inclined with the axes then $m = \pm 1$.

<u>Practice Problems :</u>

1. Find the angle between the parabolas $y^2 = 4ax$ and $x^2 = 4by$ at their point of intersection other than the origin.

2. Show that the curves $2x = y^2$ and $2xy = k$ cut at right angles if $k^2 = 8$.

3. Find the point on the curve $y - e^{xy} + x = 0$ at which we have vertical tangent.

4. Find the length of tangent, subtangent, normal and subnormal to $y^2 = 4ax$ at $(at^2, 2at)$.

5. Show that the curves

$$\frac{x^2}{a} + \frac{y^2}{b} = 1 \text{ and } \frac{x^2}{a_1} + \frac{y^2}{b_1} = 1$$ **will cut orthogonally if $a - b = a_1 - b_1$.**

6. Let P be any point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$. Then find the length of the segment of the tangent between the coordinate axes.

7. Find the value of 'c' such that the curves $x^2 - 4y^2 + c = 0$ and $y^2 = 4x$ will intersect orthogonally.

[Answers : (3) (1, 0) (4) $2at\sqrt{1+t^2}$, $2a\sqrt{1+t^2}$, $2at^2$, $2a$ (6) a (7) $c \le 64$]

C5. Errors and approximations

Let $y = f(x)$. If δx is an error in x then the corresponding error in y is δy. These small values δx and δy are called differentials. Then $\delta y = f'(x).\delta(x)$.

(i) Absolution Error : δx is called an absolute error in x.

(ii) Relative Error : $\frac{\delta x}{x}$ is called the relative error.

(iii) Percentage Error : $\left(\frac{\delta x}{x} \times 100\right)$ is called the percentage error.

<u>Practice Problems :</u>

1. Find the approximate value of :

(i) $(127)^{1/3}$ (ii) $\sqrt{26}$

2. The time of a complete oscillation of a simple pendulum of length *l* is given by the relation $T = 2\pi.\sqrt{\frac{l}{g}}$, where g is a constant. By what per cent should the length be changed in order to correct a loss of 2 minutes per day ?

[Anssers : (2) $\frac{100}{361}\%$]

Miscellaneous Problems :

1. If an triangle ABC, the side c and the angle C remains constant while the remaining elements are changed slightly, show that $\frac{da}{\cos A}+\frac{db}{\cos B}=0$.

2. If x + y = k is normal to $y^2 = 12x$ then k is

(a) 3 (b) 9 (c) –9 (d) –3

Ans. : b

3. The line x/a + y/b = 1 touches the curve $y = be^{-x/a}$ at the point

(a) (a, b/a) (b) (–a, b) (c) (0, b) (d) none of these

Ans. : c

4. If y = 4x – 5 is a tangent to the curve $y^2 = px^3 + q$ at (2, 3), then :

(a) p = 2, q = –7 (b) p = –2, q = 7 (c) p = –2, q = –7 (d) p = 2, q = 7

Ans. : a

5. If the normal to the curve y = f (x) at the point (3, 4) makes an angle 3π/4 with the positive x-axis, then $f'(3)$ =

(a) –1 (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) 1

Ans. : d

6. The slope of the tangent to the curve $y=\int_0^x \frac{dx}{1+x^3}$ at the point where x = 2 is

(a) $\frac{1}{9}$ (b) 9 (c) $\frac{1}{3}$ (d) none of these

Ans. : a

7. The equation of the common tangent to the curves $y^2 = 8x$ and xy = –1 is

(a) 3y = 9x + 2 (b) y = 2x + 1 (c) 2y = x + 8 (d) y = x + 2

Ans. : d