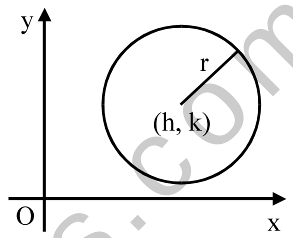
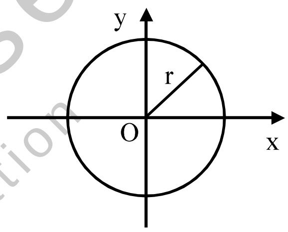
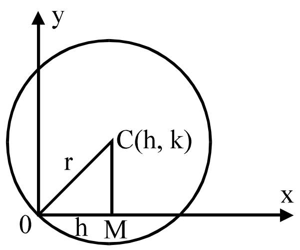
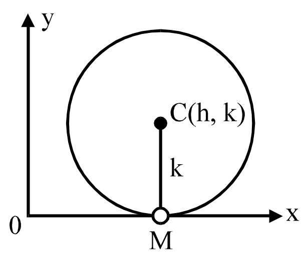
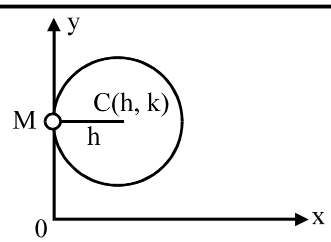
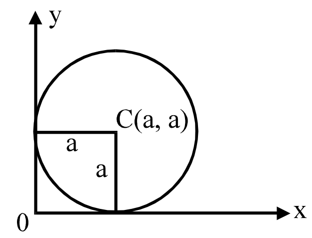
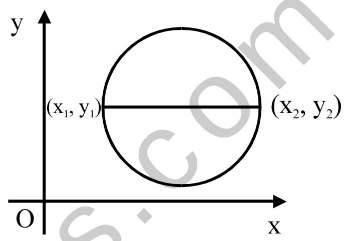
Circle

A circle is the locus of a point moving in a plane so that it always remains at a constant distance from a fixed point. The fixed point is called its center and the constant distance is called its radius.

C1 Equation of a Circle in different forms :

Forms	Equation	Figure
Standard Form	$(x - h)^2 + (y - k)^2 = r^2$ **here (h, k) is the centre and r is the radius**	
Centre at the origin	$x^2 + y^2 = r^2$	
Circle passes through the origin	$x^2 + y^2 - 2hx - 2ky = 0$	
Circle touches x-axis	$x^2 + y^2 - 2hx - 2ay + h^2 = 0$	
Circle touches y-axis	$x^2 + y^2 - 2ax - 2ky + k^2 = 0$	

Forms	Equation	Figure
Circle touches both the axes	$x^2 + y^2 - 2ax - 2ay + a^2 = 0$	
Diameter Form	$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$	
General Form	$x^2 + y^2 + 2gx + 2fy + c = 0$ **is the circle whose center is at** $(-g, -f)$ **and radius is** $\sqrt{g^2 + f^2 - c}$. **Notes :** **(i)** **If** $g^2 + f^2 - c > 0$, **then the circle is called a real circle.** **(ii)** **If** $g^2 + f^2 - c = 0$, **then the circle is called a point circle.** **(iii)** **If** $g^2 + f^2 - c < 0$, **then the circle is called an imaginary circle.** **(iv)** **The lengths of intercepts made by the circle** $x^2 + y^2 + 2gx + 2fy + c = 0$ **with x and y axes are** $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ **respectively.**	
Parametric form	**The parametric equations of** $(x - h)^2 + (y - k)^2 = r^2$ **and :** $x = h + r\cos\theta$; $y = k + r\sin\theta$; $-\pi < \theta \le \pi$ **where (h, k) is the centre, r is the radius &** θ **is a parameter**	

Practice Problems :

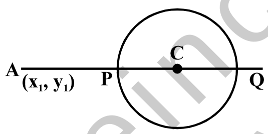
1. **Prove that the centres of the circles $x^2 + y^2 = 1$, $x^2 + y^2 + 6x - 2y - 1 = 0$ and $x^2 + y^2 - 12x + 4y = 1$ are collinear.**
2. **If the equation of two diameters of a circle are $2x + y = 6$ and $3x + 2y = 4$ and the radius is 10, find the equation of the circle.**
3. **Find the equation of the circle whose centre is (1, 2) and which passes through the point of intersection of $3x + y = 14$ and $2x + 5y = 18$.**
4. **Find the equation of the circle whose radius is 5 and the centre lies on the positive sides of x-axis at a distance 5 from the origin.**
5. **Find the equation of the circle which passes through the points (–1, 2) and (3, –2) and whose centre lies on the line $x - 2y = 0$.**
6. **Find the equation of the circle which touches both the axes and whose radius is a.**
7. **Find the equation of the circle, the end point of whose diameter are (2, –3) and (–2, 4). Find its centre and radius.**
8. **Find the equation of the circle passing through the points (1, 0), (0, 1) and (1, –2).**

[Answers : (2) $x^2 + y^2 - 16x + 20y + 64 = 0$ (3) $x^2 + y^2 - 2x - 4y - 4 = 0$ (4) $x^2 + y^2 - 10x = 0$ (5) $x^2 + y^2 - 4x - 2y - 5 = 0$ (6) $x^2 + y^2 \pm 2ax \pm 2ay - a^2 = 0$ (7) $x^2 + y^2 - y - 16 = 0$; $\left(0, \frac{1}{2}\right)$; $\frac{\sqrt{65}}{2}$ (8) $x^2 + y^2 + 2x + 2y - 3 = 0$]

C2 Position of a point with respect to a circle :

A point (x_1, y_1) lies inside, on or outside the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1 < 0$, $S_1 = 0$ or $S_1 > 0$ respectively, where $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

NOTE : The greatest and the least distance of a point A from a circle with centre C and radius r is AC + r and AC – r respectively.



Practice Problems :

1. **Does the point (–2.5, 3.5) lie inside, outside or on the circle $x^2 + y^2 = 25$?**
2. **Discuss the position of the points (1, 2) and (6, 0) with respect to the circle $x^2 + y^2 - 4x + 2y - 11 = 0$.**
3. **Find the minimum and maximum distance from the point (2, –7) to the circle $x^2 + y^2 - 14x - 10y - 151 = 0$**
4. **If the point $(\lambda, -\lambda)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$, then find range of λ.**

[Answers : (2) (1, 2) lies inside and (6, 0) lies outside (3) minimum : 2, maximum : 28 (4) $\lambda \in (-1, 4)$]

C3 Straight Line and a Circle :

Let L = 0 be a line and S = 0 be a circle. If r is the radius of the circle and p is the length of the perpendicular from the centre on the line, then

(i) $p > r$ $\Leftrightarrow$ the line does not meet the circle i.e. passes out side the circle.

(ii) $p = r$ $\Leftrightarrow$ the line touches the circle. (It is tangent to the circle)

(iii) $p < r$ $\Leftrightarrow$ the line is a sacant of the circle

(iv) $p = 0$ $\Leftrightarrow$ the line is a diameter of the circle

Note the following points :

1. The line $y = mx + c$ intersects circle $x^2 + y^2 = a^2$ in two distinct points if $c^2 < a^2(1 + m^2)$.
2. The line $y = mx + c$ touches circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$ at point $\left(\frac{-ma^2}{c}, \frac{a^2}{c}\right)$.

3. The line y = mx + c does not intersect the circle $x^2 + y^2 = a^2$ at all if $c^2 > a^2 (1 + m^2)$.

4. Equation of the pair of straight lines passing through the origin and the points of intersection of the line $lx + my + n = 0$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + (2gx + 2fy)$

$$\frac{lx+my}{-n} + c\left(\frac{lx+my}{-n}\right)^2 = 0.$$

Practice Problems :

1. **Prove that for all values of α, x sin α – y cos α = p touches the circle $x^2 + y^2 = p^2$.**
2. **Find the equation of the circle whose centre is (1, –3) and which touches the line 2x – y – 4 = 0.**
3. **Write down the equation of a circle concentric with the circle $x^2 + y^2 – 4x + 6y – 17 = 0$ and tangent to the line 3x – 4y + 7 = 0.**
4. **If the line px + qy + r = 0 touches the circle $x^2 + y^2 = a^2$ then prove that $r^2 = a^2(p^2 + q^2)$.**
5. **Find those tangents to the circle $x^2 + y^2 = 16$ which are parallel to 3x – 16y = 10.**
6. **Show that the line 7y – x = 5 touches the circle $x^2 + y^2 – 5x + 5y = 0$ and find the equation of the other parallel tangent.**

[Answers : (2) $5(x^2 + y^2) – 10x + 30y + 49 = 0$ (3) $x^2 + y^2 – 4x + 6y – 12 = 0$ (5) $3x – 16y \pm 4\sqrt{265} = 0$ (6) x – 7y – 45 = 0]

C4 The length of the intercept cut off from a line by a circle :

The intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on the co-ordinates axes are $2\sqrt{g^2 - c}$ and $\sqrt{f^2 - c}$ respectively. If

$g^2 > c$ ⇒ circle cuts the x axis at two distinct points
$g^2 = c$ ⇒ circle touches the x axis
$g^2 < c$ ⇒ circle lies completely above or below the x-axis

Similarly,

$f^2 > c$ ⇒ circle cuts the y-axis at two distinct points
$f^2 = c$ ⇒ circle touches the y-axis
$f^2 < c$ ⇒ circle lies completely above or below the y-axis

The length of the intercept cut off from the line y = mx + c by the circle $x^2 + y^2 = a^2$ is

$$2\frac{\sqrt{a^2(1+m^2)-c^2}}{(1+m^2)}.$$

Condition of Tangency : The line y = mx + c touches the circle $x^2 + y^2 = a^2$ if the length of the intercept is zero. i.e. $c = \pm a\sqrt{1+m^2}$.

Practice Problems :

1. **Find the value of λ so that the line 3x – 4y = λ, may touch the circle $x^2 + y^2 – 4x – 8y – 5 = 0$.**
2. **Find the length of the intercept on the straight line 4x – 3y – 10 = 0 by the circle $x^2 + y^2 – 2x + 4y – 20 = 0$.**
3. **Find the coordinates of the middle point of the chord which the circle $x^2 + y^2 + 4x – 2y – 3 = 0$ cuts off the line x – y + 2 = 0.**

[Answers : (1) 15, –35 (2) 10 (3) $\left(-\frac{3}{2}, \frac{1}{2}\right)$]

C5 Tangent to a Circle :

(i) Slope Form : Equations of tangents to the circle $x^2 + y^2 = r^2$ in slope form are

(a) $y = mx + r\sqrt{1+m^2}$. It touches the circle at the point $\left(\frac{-mr}{\sqrt{1+m^2}}, \frac{r}{\sqrt{1+m^2}}\right)$.

(b) $y = mx - r\sqrt{1+m^2}$. It touches the circle at the point $\left(\frac{mr}{\sqrt{1+m^2}}, \frac{-r}{\sqrt{1+m^2}}\right)$.

(ii) Point Form :

(a) Equation of the tangent to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) is $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

(b) The equation of the tangent to the circle $x^2 + y^2 = r^2$ if $c^2 = r^2(1 + m^2)$. Hence equation of tangent is $\mathbf{y = mx \pm r\sqrt{1+m^2}}$ and the point of contact is $\left(-\frac{r^2m}{c}, \frac{r^2}{c}\right)$.

Note : In general the equation of tangent to any second degree curve at point (x_1, y_1) on it can be obtained by replacing x^2 by $x\,x_1$, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$, xy by $\frac{x_1y+xy_1}{2}$ and c remains as c.

(iii) Parametric Form : The equation of a tangent to circle $x^2 + y^2 = r^2$ at $(r\cos\alpha, r\sin\alpha)$ is $x\cos\alpha + y\sin\alpha = r$.

<u>**Practice Problems :**</u>

1. Prove that the tangents to the circle $x^2 + y^2 = 25$ at (3, 4) and (4, –3) are perpendicular to each other.

2. Find the equation of tangent to the circle $x^2 + y^2 - 2ax = 0$ at the point $[a(1 + \cos\alpha), a\sin\alpha]$

3. Find the equations of the tangents to the circle $x^2 + y^2 = 9$, which

(i) are parallel to the line $3x + 4y - 5 = 0$

(ii) are perpendicular to the line $2x + 3y + 7 = 0$

(iii) make an angle of 60^0 with the x-axis.

4. Prove that the line $lx + my + n = 0$ touches the circle $(x - a)^2 + (y - b)^2 = r^2$ if $(al + bm + n)^2 = r^2(l^2 + m^2)$.

5. Show that the line $3x - 4y = 1$ touches the circles $x^2 + y^2 - 2x + 4y + 1 = 0$. Find the co-ordinates of the point of contact.

6. Show that the line $(x - 2)\cos\theta + (y - 2)\sin\theta = 1$ touches a circle for any values of θ. Find the circle.

[Answers : (2) $x\cos\alpha + y\sin\alpha = a(1 + \cos\alpha)$ (3) (i) $3x + 4y \pm 15 = 0$ (ii) $3x - 2y \pm 3\sqrt{13} = 0$ (iii) $\sqrt{3}x - y \pm 6 = 0$ (6) $x^2 + y^2 - 4x - 4y + 7 = 0$]

C6 Normal : If a line is normal/orthogonal to a circle then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is $\mathbf{y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)}$.

<u>**Practice Problems :**</u>

1. Find the equation of the normal to the circle $x^2 + y^2 = 2x$, which is parallel to the line $x + 2y = 3$.

2. Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).

3. If the radius of the circle is 5 and the equations of the two normals to the circle are $3x - 5y + 2 = 0$ and $x + 2y = 3$, find the equation of the circle.

[Answers : (1) $x + 2y - 1 = 0$ (2) $14x - 5y - 40 = 0$ (3) $x^2 + y^2 - 2x - 2y - 23 = 0$]

C7 **Length of Tangent :** The length of the tangent from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$

Note : If PQ is a length of the tangent from a point P to a given circle, then PQ^2 is called the **power of the point** with respect to a given circle.

<u>Practice Problems :</u>

1. **Find the length of tangents drawn from the point (3, –4) to the circle $2x^2 + 2y^2 - 7x - 9y - 13 = 0$.**
2. **If the length of the tangents from (f, g) to the circle $x^2 + y^2 = 6$ be twice the length of the tangent from (f, g) to circle $x^2 + y^2 + 3x + 3y = 0$ then will $f^2 + g^2 + 4f + 4g + 2 = 0$?**
3. **Show that the area of the triangle formed by the tangents from the point (4, 3) to the circle $x^2 + y^2 = 9$ and the line segment joining their points of contact is $7\frac{17}{25}$ square unit in length.**
4. **Show that the length of the tangent from any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ is $\sqrt{(c_1 - c)}$.**
5. **Find the power of point (2, 4) with respect to the circle $x^2 + y^2 - 6x + 4y - 8 = 0$.**
6. **Show that the locus of the point, the powers of which with respect to two given circles are equal, is a straight line.**

[Answers : (1) √26 (2) yes]

C8 **Combined Equation of Pair of Tangents :** Combined equation of the pair of tangents to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ drawn from the point (x_1, y_1) is $T^2 = SS_1$, where T and S_1 have their usual meaning.

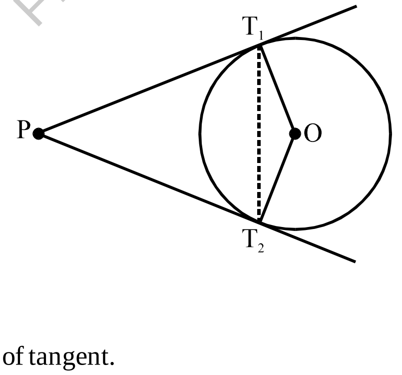
<u>Practice Problems :</u>

1. **Find the equations of the tangents to the circle $x^2 + y^2 = 16$ drawn from the point (1, 4).**
2. **The angle between a pair of tangents from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. Find the equation of the locus of the point P.**
3. **Find the equation of the tangents from the point A(3, 2) to the circle $x^2 + y^2 + 4x + 6y + 8 = 0$.**

[Answers : (1) $8x + 15y = 68$ (2) $(x + 2)^2 + (y - 3)^2 = 4$ (3) $2x - y - 4 = 0$ and $x - 2y + 1 = 0$]

C9A **Chord of Contact :** If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is :

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$



Note :

Here R = radius; L = length of tangent.

(a) Chord of contact exists only if the point 'P' is not inside.

(b) Length of chord of contact $\mathbf{T_1T_2 = \frac{2LR}{\sqrt{R^2 + L^2}}}$.

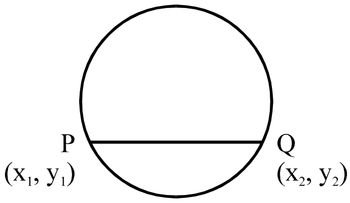
(c) Area of the triangle formed by the pair of the tangents & its chord of contact = $\frac{\mathbf{RL^3}}{\mathbf{R^2+L^2}}$

(d) Tangent of the angle between the pair of tangents from $(x_1, y_1) = \left(\frac{\mathbf{2RL}}{\mathbf{L^2-R^2}}\right)$

(e) Equation of the circle circumscribing the triangle PT_1T_2 is $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$

C9B Equation of a Chord Joining Two Point On The Curve :

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be a two point on a circle $x^2 + y^2 + 2gx + 2fy + c = 0$



Then the equation of chord joining any two point on the curve is given by

$$(x - x_1)(x_1 + x_2 + 2g) + (y - y_1)(y_1 + y_2 + 2f) = 0$$

The equation of the chord when $P = (-g + r\cos\theta_1, -f + r\sin\theta_1)$ and $Q = (-g + r\cos\theta_2, -f + r\sin\theta_2)$ are on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$$(x+g)\cos\left(\frac{\theta_1+\theta_2}{2}\right)+(y+f)\sin\left(\frac{\theta_1+\theta_2}{2}\right)=r\cos\left(\frac{\theta_1-\theta_2}{2}\right)\text{ where } r=\sqrt{g^2+f^2-c}.$$

C9C Equation of a Chord in Terms of its Mid Point : Let (x_1, y_1) be the mid point of a chord of the circle S $\equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord is $T = S_1$.

<u>Practice Problems :</u>

1. **Find the condition that chord of contact of any external point (h, k) to the circle $x^2 + y^2 = a^2$ should subtend right angle at the centre of the circle.**
2. **The chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$. Show that a, b, c are in G.P.**
3. **Find the equation of the chord of $x^2 + y^2 - 6x + 10y - 9 = 0$ which is bisected at (–2, 4).**
4. **Find the middle point of the chord intercepted on line $lx + my + n = 0$ by the circle $x^2 + y^2 = a^2$.**
5. **Through a fixed point (h, k), secants are drawn to the circle $x^2 + y^2 = r^2$. Show that the locus of mid point of the portions of secants intercepted by the circle is $x^2 + y^2 = hx + ky$.**
6. **Find the locus of middle points of chords of the circle $x^2 + y^2 = a^2$, which subtend right angle at the point (c, 0).**

[Answers : (1) $h^2 + k^2 = 2a^2$ (3) $5x - 9y + 46 = 0$ (6) $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$]

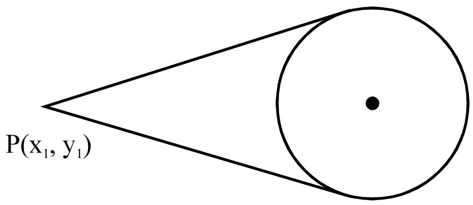
C10 Director Circle : The locus of the point of intersection of two perpendicular tangents to a circle is called the **Director Circle.**

Equation of director circle : Let the circle $x^2 + y^2 = a^2$. Then equation of the pair of tangents to a circle from a point (x_1, y_1) is $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$. If this represent a pair of perpendicular lines then.

The essential condition is coefficient of x^2 + coefficient of $y^2 = 0$

i.e. $(x_1^2 + y_1^2 - a^2 - x_1^2) + (x_1^2 + y_1^2 - a^2 - y_1^2) = 0$

$\Rightarrow$ $x_1^2 + y_1^2 = 2a^2$



Hence the equation of director circle is $x^2 + y^2 = 2a^2$

$\therefore$ Director circle is a concentric to the given circle but whose radius is $\sqrt{2}$ times the radius of the given circle.

Practice Problems :

1. **If two tangents are drawn from a point on the circle $x^2 + y^2 = 50$ to the circle $x^2 + y^2 = 25$ then find the angle between the tangents.**

[Answers : (1) 90^0]

C11 Family of Circles :

(i) Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a circle and $L \equiv ax + by + k = 0$ be a line intersecting $S = 0$, then the equation of the family of circles passing through the intersection of the given circle and the line is $S + \lambda L = 0$, where λ is a parameter.

(ii) Let $S = 0$ and $S' = 0$ be two intersecting circles. Then equation of the family of circles passing through points of intersection of $S = 0$ and $S' = 0$ is $S + \lambda S' = 0$, where $\lambda \neq -1$. $S + \lambda (S - S') = 0$ also represent a family of circles through the point of intersection of the circles $S = 0$ and $S' = 0$.

(iii) Equation of the family of circles each member of which touches the line $ax + by + c = 0$ at a point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + \lambda (ax + by + c) = 0$.

(iv) Equation of the family of circles each member of which passes through the points (x_1, y_1) and (x_2, y_2) is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$.

(v) Equation of the family of circles each member of which touches the x-axis is $x^2 + y^2 + 2gx + 2ay + g^2 = 0$. If $a > 0$, then the circles lie below x-axis.

(vi) Equation of the family of circles each member of which touches the y-axis is $x^2 + y^2 + 2ax + 2fy + f^2 = 0$. If $a < 0$, then the circles lie on the right of y-axis.

(vii) Equation of the family of circles each member of which touches the both axes and lies in the first quadrant is $x^2 + y^2 - 2ax - 2ay + a^2 = 0$, where $a > 0$.

(viii) Equation of the family of circles each member of which touches both the axes and lies in the second quadrant is $x^2 + y^2 + 2ax - 2ay + a^2 = 0$, where $a > 0$.

(ix) Equation of the family of circles each member of which touches both the axes and lies in the third quadrant is $x^2 + y^2 + 2ax + 2ay + a^2 = 0$, where $a > 0$.

(x) Equation of the family of circles each member of which touches both the axes and lies in the fourth quadrant is $x^2 + y^2 - 2ax + 2ay + a^2 = 0$, where $a > 0$.

Practice Problems :

1. **Find the equation of the circle passing through (1, 1) and the points of intersection of the circles $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$.**

2. **Find the equation of the circle passing through the point of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$, $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line $y = x$.**

3. **Find the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x - 12y + 40 = 0$ and whose radius is 4.**

4. **Find the equation of the circle through points of intersection of the circle $x^2 + y^2 - 2x - 4y + 4 = 0$ and the line $x + 2y = 4$ which touches the line $x + 2y = 0$.**

[Answers : (1) $4x^2 + 4y^2 + 30x - 13y - 25 = 0$ (2) $7x^2 + 7y^2 - 10x - 10y - 12 = 0$ (3) $2x^2 + 2y^2 - 18x - 22y + 69 = 0$ and $x^2 + y^2 - 2y - 15 = 0$ (4) $x^2 + y^2 - x - 2y = 0$]

C12 **Two Circles :** Let there be two circles with centers at C_1 and C_2 and radii r_1 and r_2 respectively, then

(i) Two circles are exterior to each other if $r_1 + r_2 < C_1C_2$

(ii) Two circles touch each other externally if $r_1 + r_2 = C_1C_2$

(iii) Two circles intersect each other in two points if $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

(iv) Two circles touch each other internally if $|r_1 - r_2| = C_1C_2$

(v) One circle is interior to the other if $C_1C_2 < |r_1 - r_2|$.

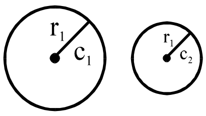
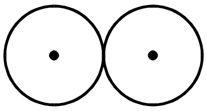
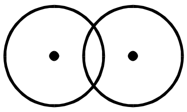
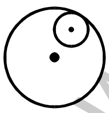
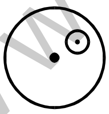
Practice Problems :

1. **Examine if the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally.**

2. **Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.**

[Answers : (1) internally]

C13 **Common Tangents to two Circles :**

	Case	Number of Tangents	Condition
(i)		**4 common tangents (2 direct and 2 transverse)**	$r_1 + r_2 < c_1 c_2$
(ii)		**3 common tangents**	$r_1 + r_2 = c_1 c_2$
(iii)		**2 common tangents**	$\lvert r_1 - r_2 \rvert < c_1 c_2 < r_1 + r_2$
(iv)		**1 common tangent**	$\lvert r_1 - r_2 \rvert = c_1 c_2$
(v)		**No common tangent**	$c_1 c_2 < \lvert r_1 - r_2 \rvert$

(Here c_1c_2 is distance between centres of two circles)

Notes :

(i) The direct common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.

(ii) Length of an external (or direct) common tangent & internal (or transverse) common tangent to the two circles are given by :

$L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}$ & $L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$, where d = distance between the centres of the two circles and r_1, r_2 are the radii of the two circles. Note that length of internal common tangent is always less than the length of the external or direct common tangent.

Practice Problems :

1. **Find all the common tangents to the circle**

   $x^2 + y^2 – 2x – 6y + 9 = 0$ and $x^2 + y^2 + 6x – 2y + 1 = 0$

2. **Show that the common tangents to the circles $x^2 + y^2 – 6x = 0$ and $x^2 + y^2 + 2x = 0$ form an equilateral triangle.**

C14 Radical Axis : Radical axis of two circles is the locus of a point from which tangents drawn to circles are of equal lengths. Radical axis of circles $S_1 = 0$ and $S_2 = 0$ is the line $S_1 – S_2 = 0$ i.e. $2(g_1 – g_2)x + 2(f_1 – f_2)y + (c_1 – c_2) = 0$.

The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles.

Notes :

(i) If the two circles intersect each other, then their common chord is the radical axis.

(ii) If the two circles touch each other, then their common tangent at their point of contact is the radical axis.

(iii) Radical axis is always perpendicular to the line joining the centres of the two circles.

(iv) Radical axis will pass through the mid point of the line joining the centres of the two circles only if the two circles have equal radii.

(v) Radical axis bisects a common tangent between the two circles.

(vi) A system of circles, every two which have the same radical axis, is called a coaxal system.

(vii) Pairs of circles which do not have radical axis are concentric.

Practice Problems :

1. **Prove that the length of the common chord of the two circles :**

   $(x – a)^2 + (y – b)^2 = c^2$ and $(x – b)^2 + (y – a)^2 = c^2$ is $\sqrt{4c^2 - 2(a-b)^2}$.

2. **Find the circle whose diameter is the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.**

3. **If two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other, then $f'g' = fg'$.**

4. **Show that the difference of the squares of the tangents to two coplanar circles from any point P in the plane of the circles vaies as the perpendicular from P on their radical axis. Also prove that the locus of a point such that the difference of the squares of the tangents from it to two given circles is constant is a line parallel to their radical axis.**

5. **Find the radical centre of circles $x^2 + y^2 + 3x + 2y + 1 = 0$, $x^2 + y^2 – x + 6y + 5 = 0$ and $x^2 + y^2 + 5x – 8y + 15 = 0$. Also find the equation of the circle cutting them orthogonally.**

6. **Find the radical centre of three circles described on the three sides $4x – 7y + 10 = 0$, $x + y – 5 = 0$ and $7x + 4y – 15 = 0$ of a triangle as diameters.**

7. **Prove that the tangents from any point of a fixed circle of co-axial system to two other fixed circles of the system are in a constant ratio.**

[Answers : (2) $2x^2 + 2y^2 + 2x + 6y + 1 = 0$ (3) $gf' = g'f$ (5) $x^2 + y^2 – 6x – 4y – 14 = 0$ (6) (1, 2)]

C15 Angle of Intersection of Two Circles : Let θ be the angle of intersection of two circles whose centers are at C_1 and C_2 and their radii are r_1 and r_2 respectively, then $\cos(\pi - \theta) = \dfrac{r_1^2 + r_2^2 - C_1C_2^2}{2r_1r_2}$.

Notes :

(i) If $\theta = 90^0$, then the circles are said to be Orthogonal and then $r_1^2 + r_2^2 = C_1C_2^2$.

(ii) Circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ are orthogonal iff $2gg' + 2ff' = c + c'$.

Practice Problems :

1. Find the angle between the circles

S : $x^2 + y^2 - 4x + 6y + 11 = 0$ and S' : $x^2 + y^2 - 2x + 8y + 13 = 0$.

2. Show that the circles $x^2 + y^2 - 6x + 4y + 4 = 0$ and $x^2 + y^2 + x + 4y + 1 = 0$ are orthogonal to each other.

3. Find the equation of the circle which cuts the circle $x^2 + y^2 + 5x + 7y - 4 = 0$ orthogonally, has its centre on the line $x = 2$ and passes through the point (4, –1).

4. Find the equation of the two circles which intersect the circles

$x^2 + y^2 - 6y + 1 = 0$ and $x^2 + y^2 - 4y + 1 = 0$, orthogonally and touch the line $3x + 4y + 5 = 0$.

5. Prove that the two circles, which pass through (0, a) and (0, –a) and touch the line $y = mx + c$, will cut orthogonally, if $c^2 = a^2(2 + m^2)$.

6. Find the equation of the circle which cuts orthogonally each of the three circles given below :

$x^2 + y^2 - 2x + 3y - 7 = 0$, $x^2 + y^2 + 5x - 5y + 9 = 0$ and $x^2 + y^2 + 7x - 9x + 29 = 0$.

[Answers : (1) $\theta = 135^0$ (3) $x^2 + y^2 - 4x + 2y + 1 = 0$ (4) $x^2 + y^2 - 1$ and $4x^2 + 4y^2 - 15x - 4 = 0$ (6) $x^2 + y^2 - 16x - 18y - 4 = 0$]

C16 Pole and Polar :

(i) If through a point P in a plane of the circle there be drawn any straight line to meet the circle in Q and R, the locus of the point of intersection of the tangents at Q & R is called the Polar of the point P; also P is called the Pole of the Polar.

(ii) The equation to the polar of a point $P(x_1, y_1)$ w.r.t. the circle $x^2 + y^2 = a^2$ is given by $xx_1 + yy_1 = a_2$, & if the circle is general then the equation of the polar becomes $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ i.e. T = 0. Note that if the point (x_1, y_1) be on the circle then the tangent & polar will be represented by the same equation. Similarly if the point (x_1, y_1) be outside the circle then the chord of contact & polar will be represented by the same equation.

(iii) Pole of a given line $Ax + By + C = 0$ w.r.t. circle $x^2 + y^2 = a^2$ is $\left(-\frac{Aa^2}{C}, -\frac{Ba^2}{C}\right)$.

(iv) If the polar of a point P pass through a point Q then the polar of Q passes through P.

(v) Two lines L_1 & L_2 are conjugate of each other if Pole of L_1 lies on L_2 & vice versa. Similarly two points P & Q are said to be conjugate of each other if the polar of P passes through Q & vice-versa.

Practice Problems :

1. Find the locus of the pole of the line $lx + my + n = 0$ with respect to the circle which touches y-axis at the origin.

2. The pole of a straight line with respect to the circle $x^2 + y^2 = a^2$ lies on the circle $x^2 + y^2 = 9a^2$. Prove that the straight line touches the circle $x^2 + y^2 = a^2/9$.

3. Prove that the polar of a given point with respect to any one of the circles $x^2 + y^2 - 2kx + c^2 = 0$, where k is a variable, always passes through a fixed point, whatever be the value of k.

4. Show that the polars of the point (1, –2) with respect to the circles $x^2 + y^2 + 6x + 5 = 0$ and $x^2 + y^2 + 2x + 8y + 5 = 0$ coincide. Prove also that there is another point, the polars of which with respect to these circles are the same and find its co-ordiantes.

[Answers : (1) $y(lx - n) = mx^2$]

C17 SOME USEFUL HINTS & TIPS :

1. If a square whose sides are parallel to the axes is inscribed in a circle, then
   (i) x-coordinates of the center is not equal to x-coordinate of any of its vertices.
   (ii) y-coordinates of the center is not equal to y-coordinate of any of its vertices.
2. Coordinates of vertices of the square whose sides are parallel to y = x and y = –x, inscribed in a circle with centre at (x_0, y_0) and radius r are $(x_0, y_0 \pm r)$ and $(x_0 \pm r, y_0)$.
3. Let a circle passes through center of another circle and both touch each other, then radius of the inner circle is half the radius of the other, further center of the inner circle lies on the join of the center of the bigger circle and their point of contact.
4. The locus of mid point on chords of a circle subtending a constant angle at the centre of the circle is always a circle concentric to the given circle.
5. Radius of the circle with centre at P, drawn orthogonal to a given circle is the length of tangent drawn from P to the given circle.