Determinants

C1. Definition :

The symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called the determinant of order two. Its value is given by : $D = a_1b_2 - a_2b_1$

Expansion of Determinant :

The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order three.

Its value can be found as :

$$D = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \quad \text{or}$$

$$D = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \ldots\text{and so on.}$$

Minors :

The minor of a given element of a determinant is the determinant of the elements which remain after deleting the row and the column in which the given element stands. For example, the minor of a_1 in

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ **is** $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ and the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

Hence a determinant of order two will have "4 minors" and a determinant of order three will have "9 minors".

Cofactor :

Cofactor of the element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$; where i and j denotes the row and column in which the particular element lies.

Note that the value of a determinant of order three in terms of 'Minor' and ' Cofactor' can be written as :

$$D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \text{ or } D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \text{ and so on.}$$

C2. Properties of Determinants :

(i) The value of a determinant remains unaltered, if the rows and columns are inter-changed.

i.e. $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$$

(ii) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D' = -D$

(iii) If a determinant has all the element zero in any row or column then its value is zero,

(iv) If a determinant has any two rows (or columns) identical then its value is zero.

(v) If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number, i.e.

$$\mathbf{D} = \begin{vmatrix} \mathbf{a_1} & \mathbf{b_1} & \mathbf{c_1} \\ \mathbf{a_1} & \mathbf{b_2} & \mathbf{c_2} \\ \mathbf{a_3} & \mathbf{b_3} & \mathbf{c_3} \end{vmatrix} \text{ and } \mathbf{D'} = \begin{vmatrix} \mathbf{Ka_1} & \mathbf{Kb_1} & \mathbf{Kc_1} \\ \mathbf{a_2} & \mathbf{b_2} & \mathbf{c_2} \\ \mathbf{a_3} & \mathbf{b_3} & \mathbf{c_3} \end{vmatrix}. \text{ Then } \mathbf{D'} = KD$$

(vi) If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants, i.e.

$$\begin{vmatrix} \mathbf{a_1+x} & \mathbf{b_1+y} & \mathbf{c_1+z} \\ \mathbf{a_2} & \mathbf{b_2} & \mathbf{c_2} \\ \mathbf{a_3} & \mathbf{b_3} & \mathbf{c_3} \end{vmatrix} = \begin{vmatrix} \mathbf{a_1} & \mathbf{b_1} & \mathbf{c_1} \\ \mathbf{a_2} & \mathbf{b_2} & \mathbf{c_2} \\ \mathbf{a_3} & \mathbf{b_3} & \mathbf{c_3} \end{vmatrix} + \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{a_2} & \mathbf{b_2} & \mathbf{c_2} \\ \mathbf{a_3} & \mathbf{b_3} & \mathbf{c_3} \end{vmatrix}$$

Practice Problems :

1. **Show that** $\begin{vmatrix} \alpha & \beta & \gamma \\ \theta & \phi & \psi \\ \lambda & \mu & \nu \end{vmatrix} = \begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \psi \end{vmatrix}$.

2. **If a, b, c are positive and are the pth, qth and rth terms respectively of a G.P., Show without expanding that**

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$$

3. **Without expanding the determinant, show that (a + b + c) is a factor of the following determinant :** $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

C3. Operation on Determinants :

1. Summation of Determinants

Let $\Delta(\mathbf{r}) = \begin{vmatrix} \mathbf{f(r)} & \mathbf{g(r)} & \mathbf{h(r)} \\ \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \\ \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \end{vmatrix}$ where a_1, a_2, a_3, b_1, b_2, b_3 are constant independent of r, then

$$\sum_{r=1}^{n} \Delta(\mathbf{r}) = \begin{vmatrix} \sum_{r=1}^{n} \mathbf{f(r)} & \sum_{r=1}^{n} \mathbf{g(r)} & \sum_{r=1}^{n} \mathbf{h(r)} \\ \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \\ \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \end{vmatrix}$$

Here function of r can be the elements of only row or column. None of the elements other than that row or column should be dependent on r. If more than one column or row have elements dependent on r then first expand the determinant and then find the summation.

2. Multiplication of Two Determinants :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} a_1l_1+b_1l_2+c_1l_3 & a_1m_1+b_1m_2+c_1m_3 & a_1n_1+b_1n_2+c_1n_3 \\ a_2l_1+b_2l_2+c_2l_3 & a_2m_1+b_2m_2+c_2m_3 & a_2n_1+b_2n_2+c_2n_3 \\ a_3l_1+b_3l_2+c_3l_3 & a_3m_1+b_2m_3+c_3m_3 & a_3n_1+b_3n_2+c_3n_3 \end{vmatrix}$$

We have multiplied here rows by rows but we can also multiply rows by column, columns by rows and columns by columns.

3. Integration of a determinant

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ where a_1, b_1, c_1, a_2, b_2, c_2 are constant independent of x. Hence

$$\int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)dx \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

If more than one row or one column are function of x then first expand the determinant and then integrate it.

4. Differentiation of Determinant :

Let $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$

then $\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$

Practice Problems :

1. **If $\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$, then prove that $\sum_{a=1}^{n} \Delta_a$ is equal to 0**

2. **If $f(x) = \begin{vmatrix} \sin^5 x & \ln \sin x & \dfrac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} \\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k \\ \dfrac{8}{15} & \dfrac{\pi}{2}\ln\left(\dfrac{1}{2}\right) & \dfrac{\pi}{4} \end{vmatrix}$. Then prove that the value of $\int_0^{\pi/2} f(x)\,dx$ is zero.**

3. **Let n be a positive integer and**

$$\Delta_r = \begin{vmatrix} 2r-1 & {}^nC_r & 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}, \text{ then prove that : } \sum_{r=0}^{n} \Delta_r = 0.$$

4. **If** $y = \dfrac{u}{v}$**, where u and v are functions of x, show that** $v^3 \dfrac{d^2y}{dx^2} = \begin{vmatrix} u & v & 0 \\ u' & v' & v \\ u'' & v'' & 2v' \end{vmatrix}$.

C4. **Cramer's Rule :** This rule is used to analyse the system of linear equations.

1. **For two Variables :** Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

(a) If the above equations have definite and unique solution then they are consistent (i.e., they are intersecting lines)

(b) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ then given equations are inconsistent or there is no solution (i.e., they are parallel lines).

(c) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ then given equations are dependent and there is infinite solution (i.e., they are identical lines)

2. **For three variables**

Let $a_1x + b_1y + c_1z = d_1$........(I)

$a_2x + b_2y + c_2z = d_2$........(II)

$a_3x + b_3y + c_3z = d_3$........(III)

Then $\mathbf{x = \dfrac{D_1}{D}, Y = \dfrac{D_2}{D}, Z = \dfrac{D_3}{D}}$

where $\mathbf{D} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \mathbf{D_1} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}; \mathbf{D_2} = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ **and** $\mathbf{D_3} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Consistency of a system of Equations

(a) If $D \neq 0$ and atleast one of $D_1, D_2, D_3 \neq 0$ then the given system of equations are consistent and have unique non trivial solution.

(b) If $D \neq 0$ and $D_1 = D_2 = D_3 = 0$ then the given system of equations are consistent and have trivial solution only.

(c) If $D = D_1 = D_2 = D_3 = 0$ then the given system of equations have either infinite solutions or no solution.

- Cofactors of D
  - All zeros
    - All the cofactors of D_1, D_2, D_3 are zero, then system will have infinite solutions
    - Cofactors of atleast one of D_1, D_2 and D_3 is non zero, then system will have no solution.
  - Atleast one non zero, then system will have infinite solutions

(d) If D = 0 but atleast one of D_1, D_2, D_3 is not zero then the equation are inconsistent and have no solution.

(e) If a given system of linear equations have only zero solution for all its variables then the given equations are said to have TRIVIAL SOLUTION.

(iv) Three equations in two variables :

If x and y are not zero, then condition for $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and

$a_3x + b_3y + c_3 = 0$ to be consistent in x and y is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Practice Problems :

1. **Show that the following system of equations is inconsistent : 2x + y = 3, 4x + 2y = 5.**
2. **Determine the values of λ for which the following system of equations fail to have a unique solution : λx + 3y – z = 1, x + 2y + z = 2, –λx + y + 2z = –1**
   Does it have any solution for this value of λ ?
3. **For what values of a and b, the system of equations 2x + ay + 6z = 8, x + 2y + bz = 5, x + y + 3z = 4 has**
   (i) a unique solution (ii) infinitely many solutions (iii) no solution.
4. **If the system of equations :**
   x = cy + bz, y = az + cx, z = bx + ay
   has a non-trivial solution, show that $a^2 + b^2 + c^2 + 2abc = 1$
5. **Using Cramer's Rule, solve the following system of linear equations :**
   x + y + z = 1
   ax + by + cz = p
   $a^2x + b^2y + c^2z = p^2$.
6. **Solve the following system of equations by Cramer's rule :**
   2x + 3y = 10
   x + 6y = 4.

Answers : **(2) –7/2**

C5. Application of Determinants :

(i) Area of a triangle whose vertices are (x_r, y_r); r = 1, 2, 3 is $D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. If D = 0 then the three points are collinear.

(ii) Equation of a straight line passing through (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(iii) The lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ are concurrent if

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Practice Problems :

1. **If the points (a_1, b_1), (a_2, b_2) and $(a_1 + a_2, b_1 + b_2)$ are collinear, show that $a_1b_2 = a_2b_1$.**
2. **Using determinants, find the value of k so that the points (k, 2 – 2 k), (– k + 1, 2k) and (– 4 – k, 6 – 2k) may be collinear.**

Answers : (2) $k = -1, \frac{1}{2}$