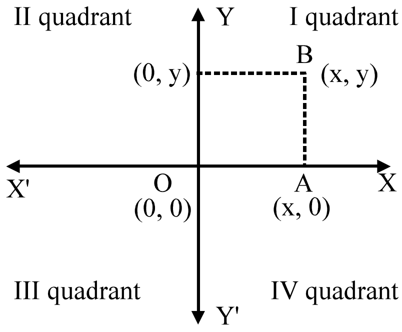
Straight line

C1 **Cartesian Co-ordinates :** Let XX' and YY' be two perpendicular straight lines drawn through any fixed point O (origin) in the plane of the paper. Then



Axis of x : The horizontal line XX'.

Axis of y : The vertical line YY'.

Co-ordinate axes : X-axis and Y-axis together are called axes of co-ordinates.

OA = PB = x (is known as **abscissa**)

AB = OP = y (is known as **ordinate**)

Note :

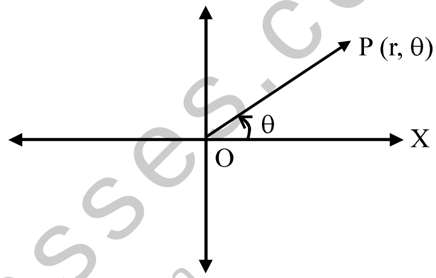
(i) Co-ordinates of the origin is (0, 0)

(ii) y co-ordinate of x-axis is zero i.e. the co-ordinate of any point on x-axis is (x, 0)

(iii) x co-ordinate of y-axis is zero i.e. the co-ordinate of any point on y-axis is (0, y)

(iv) I quadrant : $x > 0, y > 0$, II quadrant : $x < 0, y > 0$, III quadrant : $x < 0, y < 0$, IV quadrant : $x > 0, y < 0$

C2 **Polar Co-ordinates system :**

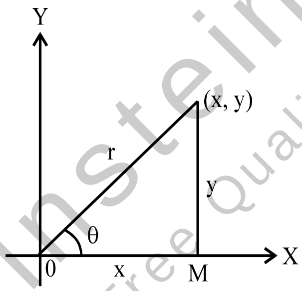


Let OX be any fixed line which is usually called the initial line and let O be a fixed point on it. This point O is called pole.

If distance of any point P from the pole O is 'r' and $\angle XOP = \theta$, then (r, θ) are called the polar co-ordinates of a point P. "r" is also known as radius vector and θ as the vectorial angle.

The radius vector is positive if it be measured from the origin O along the line bounding the vectorial angle; if measured in the opposite direction it is negative.

C3 **Relation between Polar Co-ordinates and Cartesian Co-ordinates**



$x = r \cos \theta$ and $y = r \sin \theta$

C4 **Distance Formula :** In Cartesian co-ordinate the distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

In polar co-ordinates the distance between two points $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ is given by

$$PQ = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$$

Note :

(i) If the distance between two points is given then use ± sign.

(ii) **Condition of Collinearity using the distance formula :** For three points to be collinear, prove that the sum of the distances between two points is equal to the third pair of points.

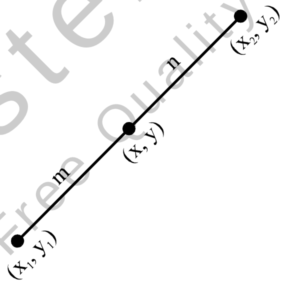
Practice Problems :

1. **Find the distance between the points (i) (a cos α, a sin α) and (0, 0) (ii) (a cos α, a sin α) and (a cos β, a sin β)**
2. **Using the distance formula, prove that the points A(–2, 3), B(1, 2) and C(7, 0) are collinear.**
3. **Prove that the points (i) (0, 5), (–2, –2), (5, 0) and (7, 7) are the vertices of a rhombus (ii) (1, –2), (3, 6), (5, 10) and (3, 2) are the vertices of a parallelogram (iii) (0, –1), (6, 7), (–2, –3) and (8, 3) are the vertices of a rectangle.**
4. **The base of an equilateral triangle with side 2a lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.**
5. **Find the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ when : (i) PQ is parallel to the y-axis, (ii) PQ is parallel to the x-axis.**
6. **Find a point on the x-axis, which is equidistant from the points (7, 6) and (3, 4).**
7. **Let the opposite angular points of a square be (3, 4) and (1, –1). Find the co-ordinates of the remaining angular points.**
8. **Prove that the points represented by polar co-ordinates $(0, 0), \left(3, \frac{\pi}{2}\right)$ and $\left(3, \frac{\pi}{6}\right)$ are the vertices of an equilateral triangle.**

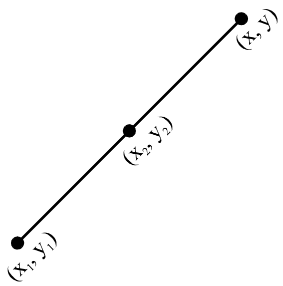
[Answers : (1) (i) |a| (ii) $2a\left|\sin\left(\frac{\alpha-\beta}{2}\right)\right|$ (4) (0, a), (0, –a) and (–√3a, 0) or (0, a), (0, –a), and (√3a, 0) (5) (i) $|y_2 - y_1|$, (ii) $|x_2 - x_1|$ (6) $\left(\frac{15}{2}, 0\right)$ (7) $\left(\frac{9}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{5}{2}\right)$]

C5 Section Formula :

1. **Internal Division :** If (x, y) are the coordinates of the point that divides the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio m : n, then $x = \frac{mx_2 + nx_1}{m+n}$ and $y = \frac{my_2 + ny_1}{m+n}$.



2. **External Division :** If (x, y) are the coordinates of the point that divides the line segment joining the points (x_1, y_1) and (x_2, y_2) externally in the ratio m : n, then $x = \frac{mx_2 - nx_1}{m-n}$ and $y = \frac{my_2 - ny_1}{m-n}$.



3. If (x, y) are the coordinates of the mid point of the line segment joining the points (x_1, y_1) and (x_2, y_2), then

$$x = \frac{x_1 + x_2}{2} \text{ and } y = \frac{y_1 + y_2}{2}.$$

Note 1 : If R (x, y) divides the join of P (x_1, y_1) and Q (x_2, y_2) in the ratio $\lambda : 1$ $(\lambda > 0)$

then $x = \dfrac{\lambda x_2 \pm x_1}{\lambda \pm 1}$; $y = \dfrac{\lambda y_2 \pm y_1}{\lambda \pm 1}$

Positive sign is taken for internal division and negative sign is taken for external division.

For finding ratio, use ratio $\lambda : 1$. If λ is positive, then divides internally and if λ is negative, then divides externally.

Note 2 : Condition of Collinearity using the concept of Section Formula : We can prove the collinearity using section formula. To understand this, take the following example :

Prove that A(5, –2), B(8, 4) and C(9, 6) are collinear points.

If the points A, B, C are collinear then any point will divide the join of other two points in a fixed ratio λ : 1. Consider the point B divides the join of the points A and C in the ratio λ : 1. Therefore calculating the value of λ from section formula

$$8 = \frac{9\lambda + 5}{\lambda + 1} \quad \Rightarrow \quad \lambda = 3$$

or $$4 = \frac{6\lambda - 2}{\lambda + 1} \quad \Rightarrow \quad \lambda = 3$$

from both equations we have getting the same value of λ. Hence the points are collinear.

<u>**Practice Problems :**</u>

1. **Find the coordinates of the point which divides the line segment joining the points A(5, –2) and B(9, 6) in the ratio 3 : 1.**
2. **Using section formula prove that the points A(–2, 3), B(1, 2) and C(7, 0) are collinear.**
3. **Without using distance formula, show that points (–2, –1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram.**
4. **Find the length of median through A of a triangle whose vertices are A(–1, 3), B(1, –1) and C(5, 1).**
5. **Find the co-ordinates of a point which divides externally the line joining (1, –3) and (–3, 9) in the ratio 1 : 3.**
6. **Find the ratio in which the point (2, y) divides the line segment joining (4, 3) and (6, 3) and hence find the value of y.**

[Answers : (1) (8, 4) (4) 5 units (5) (3, –9) (6) 1 : 2 externally, 3]

C6 Centroid or centre of gravity Triangle : The centroid of a triangle is the point of intersection of its medians. The centroid divides the medians in the ratio 2 : 1.

Centroid of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$

Incentre of a Triangle : The point of intersection of the internal bisectors of the angles of a triangle is the **incentre.** Incentre, of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) and the sides opposite to these vertices as a, b and c respectively, is given by $\left(\dfrac{ax_1 + bx_2 + cx_3}{a+b+c}, \dfrac{ay_1 + by_2 + cy_3}{a+b+c}\right)$.

Circumcentre : The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle. It is the centre of the circle which passes through the vertices of the triangle and so its distance from the vertices of the triangle and so its distance from the vertices of the triangle is the same and this distance is known as the **circum-radius** of the triangle. If angles of triangle i.e. A, B, C and vertices of triangle A (x_1, y_1), B(x_2, y_2) and C (x_3, y_3) are given, then **circumcentre** of the triangle ABC is

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right).$$

Orthocentre : The **orcthocentre** of a triangle is the point of intersection of altitudes. If angles of a triangle ABC, i.e. A, B and C and vertices of triangle are A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) given then orthocentre of the triangle ABC is

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right).$$

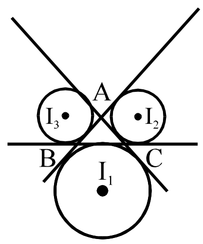
If any two lines out of three i.e., AB, BC and CA are perpendicular then orthocentre is the point of intersection of two perpendicular lines.

Excentre : A circle touches one side outside the triangle and other two extended sides then circle is known as excircle. Let ABC be a triangle then there are three excircles. With three excentres let I_1, I_2, I_3 opposite to vertices A, B & C respectively. If vertices of triangle are A(x_1, y_1), B(x_2, y_2) and C (x_3, y_3) then

$$I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right)$$

$$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c}\right)$$

$$I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c}\right)$$



Note : Let O, G and H, respectively, be the circum-centre, centroid and orthocentre of a triangle. Then O, G and H are collinear. Also G divides the line segment OH internally in the ratio 1 : 2 i.e. 2OG = GH.

Practice Problems :

1. **Two vertices of a triangle are (–1, 4) and (5, 2). If its centroid is (0, –3), find the third vertex.**

2. The vertices of a triangle are (1, a), (2, b) and $(c^2, -3)$
   - **(i) Prove that its centroid can not lie on the y-axis.**
   - **(ii) Find the condition that the centroid may lie on the x-axis.**

3. **If $\left(\frac{3}{2}, 0\right), \left(\frac{3}{2}, 6\right)$ and (–1, 6) are mid points of the sides of a triangle, then find (i) Centroid of the triangle (ii) Incentre of the triangle.**

4. **In a triangle ABC with vertices A(1, 2), B(2, 3) and C(3, 1) and $\angle A = \cos^{-1}\left(\frac{4}{5}\right), \angle B = \angle C = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$ then find the circumcentre of the triangle ABC.**

5. **If a triangle has its orthocentre at (1, 1) and circumcentre at $\left(\frac{3}{2}, \frac{3}{4}\right)$ then find the centroid.**

[Answers : (1) (–4, –15) (2) (ii) a + b = 3 (3) (i) $\left(\frac{2}{3}, 4\right)$ (ii) (1, 2) (4) $\left(\frac{11}{6}, 2\right)$ (5) $\left(\frac{4}{3}, \frac{5}{6}\right)$]

C7 **Area of a Triangle :** Area of the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \frac{1}{2}|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}|.$$

Area of a Quadrilateral : Area of the quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) is given by $\frac{1}{2}|\Delta|$, where $\Delta = \begin{vmatrix} x_3 - x_1 & y_3 - y_1 \\ x_4 - x_2 & y_4 - y_2 \end{vmatrix}$.

Area of Polygon : Area of the polygon with vertices (x_1, y_1), (x_2, y_2),....,$(x_n.y_n)$ is given by the numerical value of $\frac{1}{2}\left\{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \ldots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right\}$.

If area of triangle or quadrilateral or polygon is given, then use ± sign, positive sign indicates that the vertices are taken in anticlockwise sense & negative sign indicates that the vertices are taken in clockwise sense.

Note : Condition of Collinearity using the concept of Area :

If three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear then, area of the triangle ABC is zero i.e. $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} = 0$. Similarly if four points are collinear, then the area of the quadrilateral using these four points equals to zero.

Practice Problems :

1. **Draw a quadrilateral in the Cartesian plane, whose vertices are (–4, 5), (0, 7), (5, –5) and (–4, –2). Also, find its area.**
2. **The co-ordinates of two points A and B are (3, 4) and (5, –2) respectively. Find the co-ordinates of any point P if PA = PB and area of ΔAPB is 10.**
3. **The area of a triangle is 5 sq. units. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on y = x + 3. Find the third vertex.**

[Answers : (1) $\frac{121}{2}$ sq.unit (2) (7, 2) or (1, 0) $\left(-\frac{3}{2}, \frac{3}{2}\right)$ or $\left(\frac{7}{2}, \frac{13}{2}\right)$]

C8 **Locus :** When a point moves so as always to satisfy a given condition, or conditions, the path it trace out is called its Locus under these conditions.

If a point moves so as to satisfy any given condition it will describe some difinite curve, or locus, and there can always be found an equation between the x and y of any point on the path. This equation is called the equation to the locus or curve. This concept is illustrated by the following **example :**

Example :

A point moves so that its distance from the point (–1, 0) is always three times its distance from the point (0, 2).

Let (x, y) be any point which satisfies the given condition. We then have

$$\sqrt{(x+1)^2 + (y-0)^2} = 3\sqrt{(x-0)^2 + (y-2)^2},$$

After solving $8(x^2 + y^2) - 2x - 36y + 35 = 0$.

This is the relation between the coordinates of each, and every, point that satisfies the required condition.

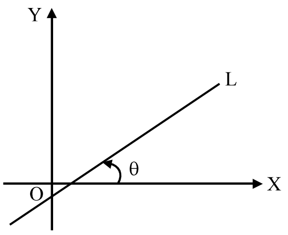
<u>Practice Problems :</u>

1. **If a point is equidistance from (6, –1) and (2, 3) then find the locus of this point.**
2. **Find the locus of the moving point such that (i) its distance from the origin is twice its distance from the y-axis (ii) sum of its distance from two fixed points (a, 0) and (–a, 0) is constant.**
3. **A variable line cuts x-axis at A, y-axis at B where OA = a, OB = b (O as origin) such that $a^2 + b^2 = 1$. Find the locus of (i) centroid of ΔOAB (ii) circumcentre of ΔOAB.**
4. **A line segment of length *l* slides with its endpoints always on x-axis and y-axis respectively. Find the locus of its midpoint.**

[Answers : (1) x – y = 3 (3) (i) $x^2 + y^2 = 1/9$ (ii) $x^2 + y^2 = 1/4$]

C9 STRAIGHT LINE : The equation be of the first degree (i.e. if it contain no products, squares, or higher powers or x and y) then locus corresponding is always a straight line.

Slope of a Straight Line : Let a straight line makes an angle $\theta \neq \pi/2$ and $0 \le \theta < \pi$, $\theta \neq \frac{\pi}{2}$ with the positive direction of x-axis. Then tanθ is called the slope of the line. It is denoted by m. If θ is 90^0, m does not exist, but the line is parallel to the y-axis. If θ = 0, then m = 0 & the line is parallel to the x-axis.



Slope of Straight Line Joining Two Points :

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points. Then the slope of AB (tan θ) is $\frac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$.

C10 Condition of collinearity of three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ using the concept of slope.

slope of AB = slope of BC = slope of AC i.e. $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$

C11 Angle Between Two Straight Lines : Let θ be the acute angle between two straight lines whose slopes are m_1 and m_2 respectively and $m_1m_2 \neq -1$, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$.

Note :

(i) If $m_1 = m_2$, then two straight line are parallel.

(ii) If $m_1m_2 = -1$, then the two lines are perpendicular to each other.

(iii) Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

(iv) Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular to each other if and only if $a_1a_2 + b_1b_2 = 0$.

<u>Practice Problems :</u>

1. **The lines x cos α + y sin α = p_1 and x cos β + y sin β = p_2 will be perpendicular if**

   (a) α = β **(b)** $|\alpha - \beta| = \frac{\pi}{2}$ **(c)** $\alpha = \frac{\pi}{2}$ **(d)** $\alpha \pm \beta = \frac{\pi}{2}$

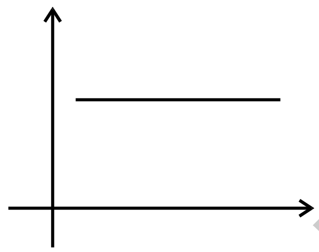
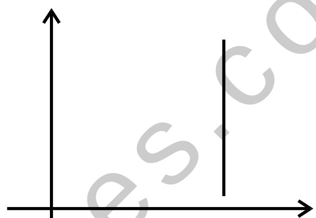
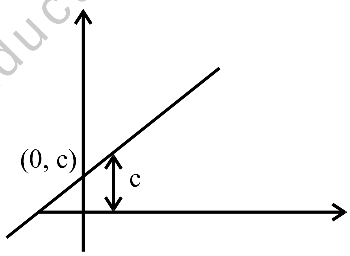
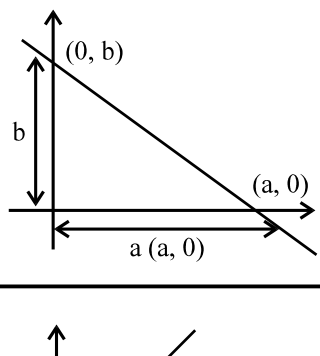
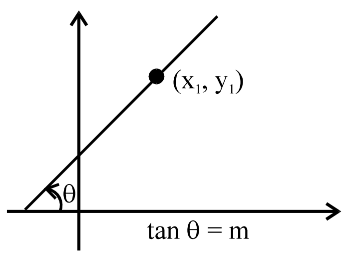
2. **The lines x + (a – 1)y + 1 = 0 and 2x + a^2y – 1 = 0 are perpendicular if**

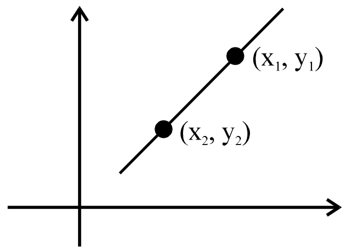
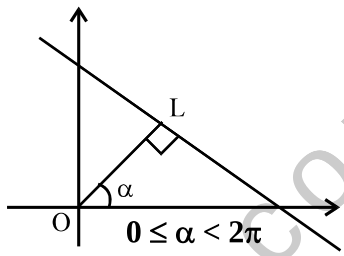
   (a) **|a| = 2** **(b)** **0 < a < 1** **(c)** **–1 < a < 0** **(d)** **a = –1**

[Answers : (1) b (2) d]

C12 Line Making an Angle α with a Given Straight Line and passing through a given point : The equation of the straight line which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are $\mathbf{y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)}$

C13 Equation of a Straight Line in different forms :

Forms	Equation	Figure
Straight line parallel to x-axis	$\mathbf{y = k}$	
Straight line parallel to y-axis	$\mathbf{x = k}$	
General Equation	$\mathbf{ax + by + c = 0}$ **slope =** $-\frac{a}{b}$ **$(b \neq 0)$**	
Slope Intercept form	**$y = mx + c$, where m is the slope and c is the intercept on y-axis. If $c = 0$ then $y = mx$ line will pass through the origin.**	
Intercept form	$\frac{x}{a} + \frac{y}{b} = 1$ **where $(a \neq 0, b \neq 0)$, where a and b are intercept on x-axis and y-axis respectively.**	
One point slope form equation of the line passing through (x_1, y_1) and slope m	$\mathbf{y - y_1 = m(x - x_1)}$	

Forms	Equation	Figure
Two point form equation of the line passing through (x_1, y_1) and (x_2, y_2) **($x_2 \neq x_2$)**	$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$	
Normal form	**$x \cos \alpha + y \sin \alpha = p$**	 **$0 \leq \alpha < 2\pi$** **OL = p (p > 0)**
Symmetric form or distance form or Parametric form	Equation of the straight line passing through a point A(x_1, y_1) and making an angle θ ($0 \leq \theta < 2\pi$) with the positive direction of x-axis is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$; $x = x_1 + r \cos \theta$, $y = y_1 + r \sin \theta$ {where r is the distance between the points A and (x, y)}.	

Practice Problems :

1. **If a straight line L is perpendicular to the line 5x – y = 1 such that area of the Δ formed by the line L and the co-ordinates axes is 5, then the equation of the line L is**

(a) $x + 5y + 5 = 0$ (b) $x + 5y \pm \sqrt{2} = 0$

(c) $x + 5y \pm \sqrt{5} = 0$ (d) $x + 5y \pm 5\sqrt{2} = 0$

2. **A (1, 3) and C(7, 5) are two opposite vertices of a square. The equation of a side through A is**

(a) $x + 2y - 7 = 0$ (b) $x - 2y + 5 = 0$

(c) $2x + y - 5 = 0$ (d) $2x - y + 1 = 0$

3. **If P (1, 0), Q (–1, 0) and R (2, 0) are three given points, then the locus of S satisfying the relation S $Q^2 + S\,R^2 = 2S\,P^2$ is**

(a) a straight line parallel to x-axis (b) a circle through the origin

(c) a circle with centre at the origin (d) a straight line parallel to y-axis

4. **A line passes through (2, 2) and is perpendicular to the line 3x + y = 3 its y-intercept is**

(a) 1/3 (b) 2/3 (c) 1 (d) 4/3

5. **One side of a rectangle lies on the line 4x + 7y + 5 = 0. Two of its vertices are (–3, 1) and (1, 1). Find the equations of other three sides.**

6. **Find the angle between the two straight lines 3x = 4y + 7 and 5y = 12x + 6 and also the equations to the two straight lines which pass through the point (4, 5) and make equal angles with the two given lines.**

7. **Find the equations of the medians of the triangle ABC whose vertices are A(2, 5), B(–4, 9) and C(–2, –1).**

8. **Find the equation of the perpendicular bisector of the line joining the points A(2, 3) and B(6, –5).**

9. **Find the equation of the straight line which passes through (3, 4) and the sum of whose intercepts on the co-ordinates axes is 14.**

10. **Show that the equation of a line passing through $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line x $\sec\theta + y\,\text{cosec}\,\theta = a$ is $x\cos\theta - y\sin\theta = a\cos 2\theta$.**

[Answers : (1) d (2) a, d (3) d (4) d (5) 7x – 4y + 25 = 0, 4x + 7y – 11 = 0, 7x – 4y –3 = 0 (6) 9x – 7y = 1, 7x + 9y = 73 (7) Equation of median AD is x – 5y + 23 = 0, Equation of median BE is 7x + 4y – 8 = 0, Equation of median CF is 8x – y + 15 = 0 (8) x – 2y – 6 = 0 (9) x + y = 7]

C14 A Point and a Straight Line :

(i) Perpendicular distance of a point (x_1, y_1) from a straight ax + by + c = 0 is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

(ii) Coordinates of the foot of perpendicular drawn from a point (x_1, y_1) to the line ax + by + c = 0 are given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}.$$

(iii) **Position of the point (x_1, y_1) relative of the line ax + by + c = 0 :** If $ax_1 + by_1 + c$ is of the same sign as c, then the point (x_1, y_1) lie on the origin side of ax + by + c = 0. But if the sign of $ax_1 + by_1 + c$ is opposite to that of c, the point (x_1, y_1) will lie on the non-origin side of ax + by + c = 0.

Two points (x_1, y_1) and (x_2, y_2) lie on the same or on the opposite side of the line ax + by + c = 0 according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are the the same or of opposite signs respectively i.e. according as $(ax_1 + by_1 + c)(ax_2 + by_2 + c)$ is positive or negative respectively.

(iv) **The ratio in which a given line divides the line segment joining two points :** If the line ax + by + c = 0 divides the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio m : n or λ : 1 then $\lambda = -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right)$ where λ equals to m/n. If A & B are on the same side of the given line then λ is negative but if A & B are on opposite sides of the given line then λ is positive or we can say if the ratio is negative then division is external and if ratio is positive then division is internal.

Practice Problems :

1. **The foot of the perpendicular from point (2, 4) upon x + y = 1 is**

(a) $\left(\frac{1}{2},\frac{3}{2}\right)$ (b) $\left(-\frac{1}{2},\frac{3}{2}\right)$ (c) $\left(\frac{4}{3},\frac{1}{2}\right)$ (d) $\left(\frac{3}{4},-\frac{1}{2}\right)$

2. **A straight line passes through a fixed point (h, k). Find the locus of the foot of the perpendicular drawn to it from origin.**

3. **In what ratio is the line joining (1, 2) and (4, 3) divided by the line joining (2, 3) and (4, 1) ?**

4. **Find the locus of a point equidistant from the lines x + y + 4 = 0 and 7x + y + 20 = 0.**

5. **Find the equation of the line passing through the point of intersection of the lines x – 3y + 1 = 0 and 2x + 5y – 9 = 0 and whose distance from the origin is $\sqrt{5}$.**

[Answers : (1) b (3) 1 : 1 (4) x – 2y = 0 (5) 2x + y – 5 = 0]

C15 Distance between Parallel Lines : Distance parallel lines ax + by + c_1 = 0 and ax + by + c_2 = 0 is equal to

$$\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}.$$

Practice Problems :

1. **The distance between the parallel lines y = 2x + 4 and 6x = 3y + 5 is**

(a) $\frac{17}{\sqrt{3}}$ (b) 1 (c) $\frac{3}{\sqrt{5}}$ (d) $\frac{17\sqrt{5}}{15}$

[Answers : (1) d]

C16 Reflection of a point about a line : Coordinates of the image of the point (x_1, y_1) in the line ax + by + c = 0 are given by $\frac{x - x_1}{a} = \frac{y - y_1}{b} = -2\frac{ax_1 + by_1 + c}{a^2 + b^2}$.

C17 Bisectors of the angles between two lines : Equations of the bisectors of the angles between two lines

$a_1x + b_1y + c_1 = 0$....................................(i)

and $a_2x + b_2y + c_2 = 0$....................................(i)

are $$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad ....(iii)$$

and $$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad ....(iv)$$

Notes : If $c_1 > 0$, $c_2 > 0$ and ($a_1b_2 \neq a_2b_1$)

(i) equation (iii) is the equation of the bisector of the angle in which the origin lies and equation (iv) is the equation of the bisector of the angle in which the origin does not lie.

(ii) further if $a_1a_2 + b_1b_2 > 0$, then equation (iii) is the equation of the obtuse angle bisector and equation (iv) is the equation of acute angle bisector.

(iii) further if $a_1a_2 + b_1b_2 < 0$, then equation (iii) is the equation of the acute angle bisector and equation (iv) is the obtuse angle bisector.

Practice Problems :

1. **Two lines passing through the point (2, 3) intersects each other at an angle of 60^0. If slope of one line is 2, find equation of the other line.**
2. **Find the equation of the right bisector of the line segment joining the points (3, 4) and (–1, 2).**
3. **Find the bisector of the angle between the lines $2x + y – 6 = 0$ and $2x – 4y + 7 = 0$ which contains the point (1, 2)**
4. **Find the equation of the bisector of the obtuse angle between the lines $3x – 4y + 7 = 0$ and $12x + 5y – 2 = 0$.**
5. **Find the bisector of acute angle between the lines $x + y – 3 = 0$ and $7x – y + 5 = 0$.**

[Answers : (1) $(\sqrt{3} + 2)x + (2\sqrt{3} – 1)y = 8\sqrt{3} + 1$ or $(\sqrt{3} – 2)x + (1 + 2\sqrt{3})y = –1 + 8\sqrt{3}$ (2) $2x + y = 5$ (3) $6x – 2y – 5 = 0$ (4) $12x + 77y – 101 = 0$ (5) $6x + 2y – 5 = 0$]

C18 Change of origin and rotation of axes : If origin is changed to O' (α, β) and axes are rotated about the new origin O' by an angle θ in the anticlockwise sense such that the new co-ordinates of P(x, y) become (x', y') then the equations of transformation will be

$$x = \alpha + x' \cos\theta – y' \sin\theta$$

$$y = \beta + x' \sin\theta + y' \cos\theta$$

Practice Problems :

1. **If the origin be shifted to the point (2, –3) and the new co-ordinate axes are parallel to the original ones, find the new co-ordinates of (4, 7).**
2. **If the origin be shifted to (1, 5), find the new equation of the line $3x + 4y = 5$.**

[Answers : (1) (2, 10) (2) $3X + 4Y + 18 = 0$]

C19 Family of Straight Lines :

1. Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ be two non-parallel lines. Then the equation of the family of straight line passing through the intersection of the lines $L_1 = 0$ and $L_2 = 0$ is $L_1 + \lambda L_2 \equiv a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$.
2. Equation of the family of straight lines parallel to the line $ax + by + c = 0$ is $ax + by + k = 0$, where k is a parameter.
3. Equation of the family of straight lines perpendicular to the line $ax + by + c = 0$ is $bx – ay + k = 0$, where k is a parameter.

Practice Problems :

1. **The family of lines $x(a + 2b) + y(a + 3b) = a + b$ passes through the point for all values of a and b. Find the point.**
2. **If $3a + 2b + 6c = 0$ the family of straight lines $ax + by + c = 0$ passes through a fixed point. Find the co-ordinates of fixed point.**
3. **If $16a^2 – 40ab + 25b^2 – c^2 = 0$, then the line $ax + by + c = 0$ passes through the points :**

   (a) (4, –5) and (–4, 5) **(b) (5, –4) and (–5, 4)**

   (c) (1, –1) and (–1, 1) **(d) none of the above**

4. **If a straight line moves so that the sum of the perpendiculars let fall on it from two fixed points (3, 4) and (7, 2) is equal to three times the perpendicular on it from a third point (1, 3) prove that there is another fixed point through which this line always passes & find its coordinates.**

[Answers : (1) (2, –1) (2) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (3) a]

C20 **Concurrent lines :** The three lines are concurrent if they meet in a point.

Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ & $a_3x + b_3y + c_3 = 0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Alternatively : If three constants A, B and C (not all zero) can be found such that

$A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) \equiv 0$, then the three straight lines are concurrent.

Note : Line $L_1 = 0$, $L_2 = 0$ and $L_1 + \lambda L_2 = 0$ are concurrent.

<u>Practice Problems :</u>

1. **If the lines x = a + m, y = –2 and y = mx are concurrent, the least value of a is**

(a) 0 **(b)** $\sqrt{2}$ **(c)** $2\sqrt{2}$ **(d)** 1

2. **The equation of the line with slope $-\frac{3}{2}$ which is concurrent with the lines 4x + 3y – 7 = 0 and 8x + 5y – 1 = 0 is**

(a) 3x + 2y – 2 = 0 **(b)** 3x + 2y – 63 = 0

(c) 2y – 3x – 2 = 0 **(d)** none of the above

3. **The number of integral values of m for which the x-co-ordinates of the point of intersection of the lines 3x + 4y = 9 and y = mx + 1 is also an integer is**

(a) 2 **(b)** 0 **(c)** 4 **(d)** 1

4. **Area of the parallelogram formed by the lines y = mx, y = mx + 1, y = nx and y = nx + 1 equals**

(a) $\frac{|m+n|}{(m-n)^2}$ **(b)** $\frac{2}{|m+n|}$ **(c)** $\frac{1}{|m+n|}$ **(d)** $\frac{1}{|m-n|}$

5. **The three lines ax + by + c = 0, bx + cy + a = 0, cx + ay + b = 0 are concurrent only when**

(a) a + b + c = 0 **(b)** $a^2 + b^2 + c^2 = 2(ab + bc + ca)$

(c) $a^3 + b^3 + c^3 = ab + bc + ca$ **(d)** none of the above

6. **The area bounded by the curves x + 2|y| = 1 and x = 0 is**

(a) 1/4 **(b)** 1/2 **(c)** 1 **(d)** 2

7. **If the lines x + a y + a = 0, b x + y + b = 0 and c x + c y + 1 = 0 (a, b and c being distinct ≠ 1) are concurrent, then the value of $\frac{a}{a-1}+\frac{b}{b-1}+\frac{c}{c-1}$ is**

(a) –1 **(b)** 0 **(c)** 1 **(d)** none of these

8. **Show that the straight lines 7x – 2y + 10 = 0, 7x + 2y + 10 = 0 and y + 2 = 0 form an isosceles triangle. Find the area.**

9. **Show that the lines (p + q)x + (p + q) y = (p – q) ; (p – q)x – (p – q)y = (p + q), px + qy = p and qx + py + q = 0 are concurrent.**

10. **Find the direction in which a straight line must be drawn through the point (1, 2), so that its point of intersection with the line x + y = 4 may be at a distance $\frac{1}{3}\sqrt{6}$ from this point.**

[Answers : (1) c (2) a (3) a (4) d (5) a (6) b (7) c]

C21 **Pair of Straight Lines :** $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin. If their slopes are m_1 and m_2, then $m_1 + m_2 = -2h/b$ and $m_1m_2 = a/b$.

Notes :
(i) The two lines are real and distinct if $h^2 - ab > 0$.
(ii) The two lines are real and coincident if $h^2 - ab = 0$.
(iii) The two lines are imaginary if $h^2 - ab < 0$.

Practice Problems :

1. **Find the separate equation of line represented by the equation $x^2 - 6xy + 8y^2 = 0$.**
2. **Find the condition that the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ should be n times the slope of the other.**
3. **Find the area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and $y = 9$.**

[Answers : (1) $x - 4y = 0$, $x - 2y = 0$ (2) $4nh^2 = ab(1 + n)^2$ (3) 27/4]

C22 **Straight Lines Perpendicular to Given Pair of Straight Lines :** The equation of the pair of straight line through the origin which are perpendicular to the lines given by $ax^2 + 2hx + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

C23 **Angle Between Pair of Straight Lines :** Let θ be the acute angle between pair of straight lines $ax^2 + 2hxy + by^2 = 0$ then $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{|a+b|}\right|$.

Notes : The condition that these lines are :

(a) At right angles to each other is $a + b = 0$ i.e. co-efficient of x^2 + co-efficient of $y^2 = 0$
(b) Coincident is $h^2 = ab$
(c) Equally inclined to the axis of x is $h = 0$ i.e. coefficient of $xy = 0$.

Practice Problems :

1. **Find the angle between the lines $(x^2 + y^2)\sin^2\alpha = (x\cos\beta - y\sin\beta)^2$**
2. **Prove that the lines $x^2 + 4xy + y^2 = 0$ and the line $x - y = 4$ form an equilateral triangle.**

[Answers : (1) 2α]

C24 **Equation of the Bisectors of the angles between the lines :** Equation of the straight lines bisecting the angles between the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

Practice Problems :

1. **If the co-ordinate axes are the angle bisectors of the pair of lines $ax^2 + 2hxy + by^2 = 0$ then prove that $h = 0$?**
2. **If the line $y = mx$ is one of the bisector of the lines $x^2 + 4xy - y^2 = 0$ then find the value of 'm' ?**

[Answers : (2) $\frac{\sqrt{5}-1}{2}, -\left(\frac{\sqrt{5}+1}{2}\right)$]

C25 **General Equation of Second Degree :** General equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

i.e. if and only if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Notes :

(i) Two lines are parallel if $G = \begin{vmatrix} h & b \\ g & f \end{vmatrix} = 0$ and $C = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = 0$.

(ii) Two lines are coincident if $G = \begin{vmatrix} h & b \\ g & f \end{vmatrix} = 0, C = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = 0$ and $A = \begin{vmatrix} b & f \\ f & c \end{vmatrix} = 0$.

(iii) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines and θ is the acute angle between them, then

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{|a + b|}\right|.$$

Note : The two lines are perpendicular to each other if and only if $a + b = 0$.
(i.e. coefficient of x^2 + coefficient of $y^2 = 0$).

Practice Problems :

1. **Prove that the angle between the line joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1}\left(\frac{2\sqrt{2}}{3}\right)$.**

2. **If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines, then the distance between them is**

(a) $\sqrt{\frac{g^2 - ac}{h^2 + a^2}}$ **(b)** $\sqrt{\frac{g^2 + ac}{a(a+b)}}$ **(c)** $2\sqrt{\frac{g^2 - ac}{h^2 + a^2}}$ **(d)** $2\sqrt{\frac{g^2 + ac}{a(a+b)}}$

[Answers : (2) c]

C26 Time Saving Tips and Shortcuts :

1. Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear

   (i) If $\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3}$, then these points lie on a straight line passing through the origin.

   (ii) If $\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3}$ is not true but $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1}$, then the points lie on a straight line not passing through the origin.

2. ΔOAB is obtuse angled if the points A and B lie in opposite quadrants.

3. In a triangle ABC

   (a) if |tan A| = |tan B| = |tan C|, then the triangle is equilateral.

   (b) if |tan A| = |tan B| < 1 or |tan A| = |tan C| < 1 or |tan B| = |tan C| < 1 then the triangle is obtuse angled isosceles.

   (c) if exactly two of |tan A|, |tan B| and |tan C| are equal to k, k > 1, then the triangle is acute angled isosceles.

4. The locus of the point equidistant from the points (a, b) and (b, a) is y = x.

5. The locus of the point equidistant from the points (a, b) and (–b, –a) is y = –x.

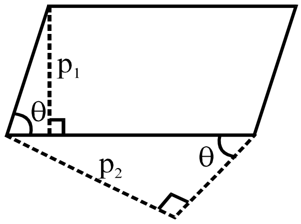
6. If $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are the sides of a triangle, then the area of the triangle is given by (without solving the vertices).

$$\Delta = \frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

when C_1, C_2, C_3 are the co-factors of c_1, c_2, c_3 in the determinant

i.e., $C_1 = a_2b_3 - a_3b_2$, $C_2 = a_3b_1 - a_2b_3$ and $C_3 = a_1b_2 - a_2b_1$.

7. **Area of Parallelogram**



Area of parallelogram ABCD $= \frac{p_1p_2}{\sin\theta}$

where p_1 and p_2 are the distances between parallel sides and θ is the angle between two adjacent sides.

Note :

(i) Area of rhombus (in case of rhombus $p_1 = p_2$) = $\frac{p_1^2}{\sin\theta}$

(ii) Area of rhombus = $\frac{1}{2}d_1d_2$, where d_1 and d_2 are the lengths of two perpendicular diagonals of a rhombus.

8. The line joining the points (x_1, y_1) and (x_1, y_2) is divided by the X-axis in the ratio $-\frac{y_1}{y_2}$ and by Y-axis in the ratio $-\frac{x_1}{x_2}$.

9. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) be respectively the vertices A, B, C of a triangle, then angle A is acute or obtuse according as $(x_1 - x_2)(x_1 - x_3) + (y_1 - y_2)(y_1 - y_3)$ is positive or negative.

10. Area of the triangle formed by the lines of the form $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ is

$$\Delta = \frac{1}{2}\left[\frac{(c_2 - c_3)^2}{m_2 - m_3} + \frac{(c_3 - c_1)^2}{m_3 - m_1} + \frac{(c_1 - c_2)^2}{m_1 - m_2}\right].$$

11. The area of a rohmbus with sides $ax \pm by + c = 0$ is $\frac{2c^2}{ab}$.

12. The area of a parallelogram formed by the lines $a_1x + b_1y + c_1 = 0$, $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + c_2 = 0$, and $a_2x + b_2y + d_2 = 0$ is

$$\frac{(c_1 - d_1)(c_2 - d_2)}{a_1b_2 - a_2b_1}.$$

13. If the lines represented by $a_1x^2 + 2h_1xy + b_1y^2 = 0$ and $a_2x^2 + 2h_2xy + b_2y^2 = 0$ are equally inclined to each other then $\frac{a_1 - b_1}{h_1} = \frac{a_2 - b_2}{h_2}$

14. Centroid of the triangle obtained by joining the middle points of the sides of a triangle is the same as the centroid of the original triangle.

15. If two vertices of an equilateral triangle are (x_1, y_1) and (x_2, y_2) then co-ordinates of the third vertex are $\left(\frac{x_1 + x_2 \pm \sqrt{3}(y_1 - y_2)}{2}; \frac{y_1 + y_2 \pm \sqrt{3}(x_1 - x_2)}{2}\right)$. The vertices of the equilateral triangle do not have integral coordinates.

16. Centroid, orthocentre, circumcentre coincides for the equilateral triangle.

17. Shifting of the origin does not alter the area of triangle.

18. The orthocentre of a triangle having vertices (α, β), (β, α) and (α, α) is (α, α).
    e.g., the orthocentre of the triangle having vertices (4, 5), (5, 4) and (4, 4) is (4, 4).

19. The image of the line $ax + by + c = 0$ about y-axis $-ax + by + c = 0$.

20. If the circumcentre and centroid of a triangle are respectively (α, β), (γ, δ) then orthocentre will be $(3\gamma - 2\alpha, 3\delta - 2\beta)$.

21. The length of the intercept cut by the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ on x-axis is $\frac{2\sqrt{g^2 - ac}}{|a|}$.

22. A line passing through (x_1, y_1) and parallel to $ax + by + c = 0$ is $a(x - x_1) + b(y - y_1) = 0$.

23. The image of the point (α, β) about the line $x = \lambda$ is $(2\lambda - \alpha, \beta)$.

24. Circumcentre of the triangle formed by the origin (0, 0) and the points (x_1, y_1) and (x_2, y_2) is $\left(\frac{\Delta_2}{\Delta_1}, \frac{\Delta_3}{\Delta_1}\right)$ where $\Delta_1 = \begin{vmatrix} 2x_1 & 2y_1 \\ 2x_2 & 2y_2 \end{vmatrix}$; $\Delta_2 = \begin{vmatrix} 2y_1 & -(x_1^2 + y_1^2) \\ 2y_2 & -(x_2^2 + y_2^2) \end{vmatrix}$; $\Delta_3 = \begin{vmatrix} -(x_1^2 + y_1^2) & 2x_1 \\ -(x_2^2 + y_2^2) & 2x_2 \end{vmatrix}$.

25. Circumcentre of the triangle formed by the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is the same as that of the triangle formed by the points (0, 0); $(x_2 - x_1, y_2 - y_1)$, $(x_3 - x_1, y_3 - y_1)$.

26. Orthocentre of the triangle formed by going the points (0, 0), (x_1, y_1), (x_2, y_2) is given by $[q(y_2 - y_1), -q(x_2 - x_1)]$ where $q = \frac{x_1x_2 + y_1y_2}{x_1y_2 - x_2y_1}$. In order to find the orthocentre of the triangle formed by (x_1, y_1), (x_2, y_2), (x_3, y_3), first shift the origin to (x_1, y_1) and then use the above tip.

27. Orthocentre of the triangle formed by the points $\left(\alpha, \frac{1}{\alpha}\right), \left(\beta, \frac{1}{\beta}\right), \left(\gamma, \frac{1}{\gamma}\right)$ **is** $\left(-\frac{1}{\alpha\beta\gamma}, -\alpha\beta\gamma\right)$.

28. The pair of lines $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$ form an equilateral triangle with the lines $ax + by + c = 0$.

29. The product perpendicular drawn from any point (α, β) on the lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{a\alpha^2 + 2h\alpha\beta + b\beta^2}{\sqrt{(a - b)^2 + 4h^2}}$.

30. Reflection of the point (α, β) about the line $y = x$ is (β, α).