Work, Energy, Power and Curvelinear Motion

WORK, ENERGY, POWER AND CURVELINEAR MOTION

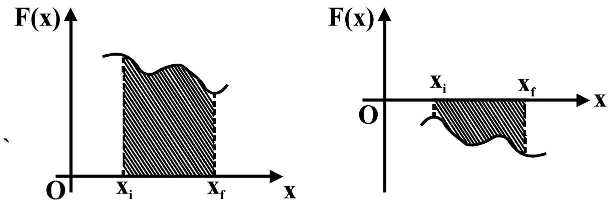
C1A Work

When a force acts on a body of particle that moves, the force can do work on the body. Mathematically, the work W_{ba} done by a force F on the particle as the particle moves from a to b is defined as $\mathbf{W_{ba}} = \int_a^b \vec{\mathbf{F}}.\mathbf{d}\vec{\mathbf{r}}$.

Hence the work done by a force is defined as the dot product of the force and the distanplacement of the point of application of force. The work done by a force can be positive, negative or zero. Work depends on frame of reference.

C1B Graphical Interpretation Of Work Done

Graphically, the work done by a variable force F(x) from an initial position x_i to final position x_f is interpreted as the area under the force-displacement curve, shown in figure



(A) Positive Work (B) Negative Work

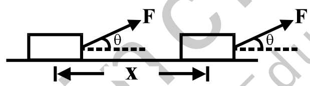
C1C Work : For Motion Along Straight Line

Case I Constant Force

Consider a body that moves along the x-axis under a constant force F then work W done by the force W =

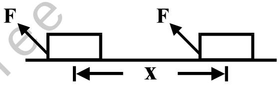
$\mathbf{W} = \vec{\mathbf{F}}.\vec{\mathbf{x}}$ = Fx cos θ

→ If $\theta < \frac{\pi}{2}$, W is positive



→ If $\theta = \frac{\pi}{2}$, W = 0

→ If $\theta > \frac{\pi}{2}$, W is negative



→ If force is in direction of displacement i.e. θ = 0, W = Fx

→ If force is in opposite to displacement i.e. θ = π, W = –Fx

Work Done By Several Constant Forces

Choose the initial and final positions of the body, and draw a free body diagram showing all the forces that act on the body. List the forces and calculate the work done by each force. Add the amounts of work done by the separate forces to find the total work done.

Case II Work Done by Variable Force

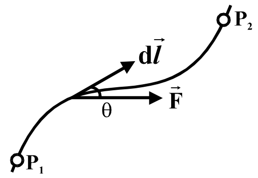
Consider a body moves along the x-axis from position x_1 to x_2 under a variable force F. Then work done by this force is $\mathbf{W} = \int_{x_1}^{x_2} \vec{\mathbf{F}}.\mathbf{d}\vec{\mathbf{x}}$.

Work done by Spring Force

The spring force is a variable force and work done by spring force to extend or compress the spring from x_1 and x_2 is given by $\mathbf{W = -\frac{1}{2}k(x_2^2 - x_1^2)}$. It does not mean that work done by spring force is always negative.

C1D Work : for Motion along Curved Path

We can generalize our definition of work further to include a force that varies in direction as well as in magnitude and a displacement that lies along a curved path. Suppose a particle moves from point P_1 to P_2 along a curve, as shown in figure :



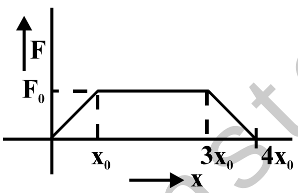
We divide the portion of the curve between these points into many infinitesimal vector displacements $\vec{dl}$. Each $\vec{dl}$ is tangent to the path at its position. Let $\vec{F}$ be the force at a typical point along the path, and let θ be the angle between $\vec{F}$ and $\vec{dl}$ at this point.

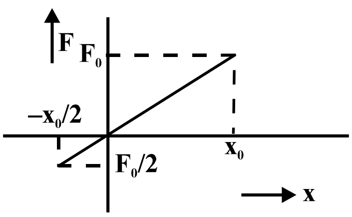
Then $\quad \mathbf{dW = \vec{F}.\vec{dl} = F\cos\theta dl}$

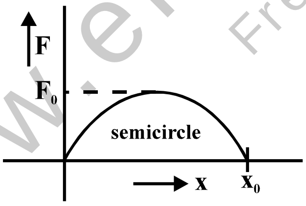
$$\Rightarrow \quad \mathbf{W = \int_{P_1}^{P_2} \vec{F}.\vec{dl}}$$

<u>Practice Problems :</u>

1. **Find the work done in the following cases :**

(a) 

(b) 

(c) 

2. **A man moves on a straight horizontal road with a block of mass 2 kg in his hand. If he moves a distance of 40 m with an accleration of 0.5 m/s^2. Calculate work done by the man on the block during motion.**

3. **A man weighing 55 kg supports a body of 20 kg on his head. Calculate work done by him if he moves a distance of 20 m (i) on a horizontal road, (ii) upon incline of 1 in 5. Take g = 10 ms^{-2}.**

4. **An elastic string of unstretched length L and force constant k is stretched by a small length x. It is further stretched by another small length y. The work done in the second stretching is**

(a) $\frac{1}{2}ky^2$ **(b)** $\frac{1}{2}k(x^2+y^2)$ **(c)** $\frac{1}{2}k(x+y)^2$ **(d)** $\frac{1}{2}ky(2x+y)$

5. **A cord is used to lower vertically a block of mass M a distance d at a constant downward acceleration of g/4. Then the work done by the cord on the block is**

(a) $\frac{Mgd}{4}$ (b) $\frac{-Mgd}{4}$ (c) $\frac{3Mgd}{4}$ (d) $\frac{-3Mgd}{4}$

6. **A tracter pulls a block, of weight 14, 700 N, a distance of 20 m along the level road. The tractor exerts a constant 5000 N force at an angle of 37^0 above the horizontal. There is a 3500 N friction force opposing the motion. The total work done by all the forces**

(a) 10 kJ (b) 20 kJ (c) 30 kJ (d) 40 kJ

[Answers : (2) 40 J (3) Zero; 3000 J (4) d (5) d (6) a]

C2 Conservative Forces

A force is conservative force if the net work it does on a particle moving around every closed path, from an initial point and then back to that point, is zero, Equivalently, it is conservative if the net work it does on a particle moving between two points does not depend on the path taken by the particle. The gravitational force and the spring force are conservatives; the kinetic frictional force is a non-conservative force. Any dissipative force is non- conservative in nature.

C3 Kinetic Energy

Kinetic energy is a scalar quantity associated with the motion of the particle. Mathematically kinetic energy of a body or particle of mass m having speed v is given by $\frac{1}{2}mv^2$. Kinetic energy depends on frame of reference.

C4 Work Energy Theorem

Work is associated with kinetic energy. In any displacement of a particle, the change in its kinetic energy equals the total work done by all the forces acting on the particle.

$$W = K_f - K_i = \Delta K$$

This result is the work - energy theorem.

→ Work energy theorem is valid even when force varies during the displacement and for all types of motions.

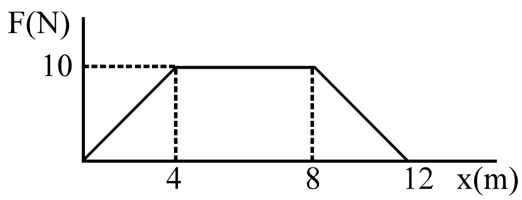
→ Work energy theorem is valid for all types of force acting on the body i.e. for conservative and non-conservative forces.

<u>**Practice Problems :**</u>

1. **The displacement x of a particle of mass m kg moving in one dimension, under the action of a force, is related to the time t by the equation $t = \sqrt{x} + 3$ where x is in metres and t is in seconds. The work done by the force in the first six seconds in joules is**

(a) 0 (b) 3 m (c) 6 m (d) 9 m

2. **A particle of mass 0.1 kg (moving along x-axis) is subjected to a force, which varies with distance (x) as shown in figure. It starts journey from rest at x = 0, its speed at x = 12 m is**



(a) 20 m/s (b) 40 m/s (c) 60 m/s (d) 80 m/s

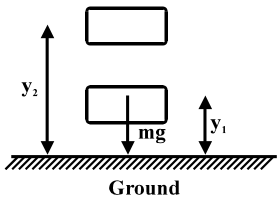
[Answers : (1) a (2) b]

C5 Potential Energy

Potential energy is energy associated with the position of a system rather than its motion. Potential energy function is defined only for conservative forces. The work done on a body by conservative forces can be expressed as a change in potential energy i.e. $dW_c = -dU$.

Gravitational Potential Energy

Work done by gravitational force during the vertical motion of a body from an initial position y_1 to final position y_2 is given $\mathbf{W_c = -mg\int_{y_1}^{y_2} dy = -mg(y_2 - y_1)}$



As $\quad dW_c = -dU \quad \Rightarrow \quad U_f - U_i = mgy_2 - mgy_1$.

Above expression shows that we can express work done by gravitational force in terms of the values of the quantity mgy at the beginning and end of the displacement. This quantity, the product of the weight mg and the position y above the reference level, is called the gravitational potential energy i.e. U = mgy.

Remember that to define the gravitational potential energy we need a reference level. Above the reference level the gravitational potential energy is positive, on the reference level the gravitational potential energy is zero whereas below the reference level the gravitational potential energy is taken as negative.

Spring Potential Energy

When a spring is elongated (or compressed) work is done against the spring force. This work done is stored in the spring as spring or elastic potential energy.

Work done in stretching or compression a spring from x_1 to x_2 is given by $\mathbf{-\frac{1}{2}k(x_2^2 - x_1^2)}$

Hence, change in spring potential energy = $\mathbf{\frac{1}{2}k(x_2^2 - x_1^2)}$.

The spring potential energy for a spring extended by amount 'x' or compressed by amount 'x' is $\frac{1}{2}kx^2$.

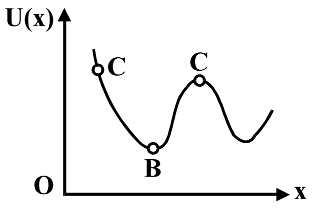
C6 Conservative Force And Potential Energy Function

A conservative force can be derived from a scalar potential energy function. The slope of the potential energy function is a measure of the magnitude of conservative force.

In one dimension, it is defined as the negative derivative of potential energy with respect to distance i.e.,

$$\mathbf{F_c = -\frac{dU}{dx}}$$

The figure shows a graph of potential energy function U(x) for one dimensional motion. Three specific points A, B and C are chosen.



At point A, a very strong force is acting because the slope is very large.

At B and C, no force acts as the slope is zero, these two are the equilibrium positions. At equilibrium, $\frac{dU}{dx} = 0$.

The point B is the position of stable equilibrium, as U is minimum and $\frac{d^2U}{dx^2} > 0$

The point C is the position of unstable equilibrium, as U is maximum and $\frac{d^2U}{dx^2} < 0$

C7 Mechanical Energy

The mechanical energy E_{mec} of a system is the sum of its kinetic K and its potential energy U :

$$E_{mec} = K + U$$

As isolated system is one in which no external force causes energy changes. If only conservatives force do work within an isolated system, then the mechanical energy E_{mec} of the system cannot change. This **principle of conservation of mechanical energy** is written as

$$K_2 + U_2 = K_1 + U_1,$$

in which the subscripts refer to different instants during an energy transfer process. This conservation principle can also be written as

$$\Delta E_{mec} = \Delta K + \Delta U = 0$$

<u>Practice Problems :</u>

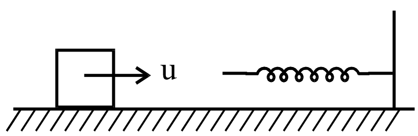
1. **The potential energy of a 2 kg particle free to move along the x-axis is given by $U(x) = \left(\frac{x}{b}\right)^4 - 5\left(\frac{x}{b}\right)^2$ J where b = 1 m. Plot this potential, identifying the extremum points. Identify the regions where particle may be found and its maximum speed given that the total mechanical energy is (i) 36 J; (ii) –4 J. Also find the force acting on the particle as a function of x.**

2. **A block of mass 5.0 kg is suspended from the end of a vertical spring which is stretched by 10 cm under the load of the block. The block is given a sharp impulse from below so that it acquires an upward speed of 2.0 m/s. How high will it rise ? Take g = 10 m/s².**

3. **A uniform chain of length L and mass M is lying on a smooth table and 1/n of its length is hanging vertically down over the edge of the table. The work required to pull the hanging part on the table is**

(a) **MgL** (b) $\frac{MgL}{n}$ (c) $\frac{MgL}{n^2}$ (d) $\frac{MgL}{2n^2}$

4. **An ideal spring of force constant k is attached to a vertical wall as shown in figure. A block of mass m is projected with speed u towards the spring. The horizontal surface is smooth. The maximum compression in the spring is**



(a) $2u\sqrt{\frac{m}{k}}$ (b) $u\sqrt{\frac{m}{k}}$ (c) $\frac{u}{2}\sqrt{\frac{m}{k}}$ (d) $\frac{u}{4}\sqrt{\frac{m}{k}}$

5. **In the above problem, the speed of the block when it compress the spring by an amount half of the maximum compression is**

(a) **u/2** (b) **u/4** (c) **u/√2** (d) **none of these**

6. **An ideal spring with spring constant k is hung from the ceiling and a block of mass M is attached to its lower end. The mass is released with the spring initially unstreched. Then the maximum extension in the spring is**

(a) $\frac{4Mg}{k}$ (b) $\frac{2Mg}{k}$ (c) $\frac{Mg}{k}$ (d) $\frac{Mg}{2k}$

[Answers : (1) (a) –3m < x < 3m, v_{max} = 5.45 m/s (b) –2m < x < –1m, 1m < x < 2m, v_{max} = 1.5 m/ (2) 20 cm (3) d (4) b (5) d (6) b]

C8 Power

The time rate of doing work is defined as power and is given by

$$(\text{Power})_{\text{instant}} = \frac{\mathbf{dW}}{\mathbf{dt}} \text{ and } (\text{Power})_{\text{av.}} = \frac{\Delta \mathbf{W}}{\Delta \mathbf{t}}$$

From $\mathbf{P} = \frac{\mathbf{dW}}{\mathbf{dt}}$, we can write P = $\vec{\mathbf{F}}.\vec{\mathbf{v}}$

<u>Practice Problems :</u>

1. **The human heart discharges 75 ml of blood at each beat against a pressure of 0.1 m of Hg. Calculate the power of the heart assuming that the pulse frequency is 80 beats per minute. Given, density of mercury = 13.6 × 10^3 kg/m^3.**
2. **A pump motor is used to deliver water at a certain rate from a given pipe. To obtain 'n' times water from the same pipe in the same time by what amount (a) the force and (b) power of the motor should be increased.**
3. **A particle is projected with speed u at an angle θ with the horizontal in a vertical plane. The instantaneous power of the particle at the instant when it at the highest point of its trajectory is**

   (a) 0 (b) mgu cosθ (c) mgu sinθ (d) mgu tanθ
4. **In the above problem, the average power during the time from point of projection to the instant when the particle is at the highest point of its trajectory is**

   (a) 0 (b) –½mgu cosθ (c) –½mgu sinθ (d) –½mgu tanθ
5. **A body is moved along a straight line by a machine delivering constant power. The distance moved by the body in time t is proportional to**

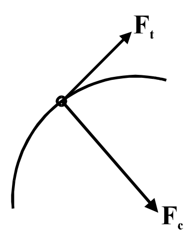
   (a) $t^{1/2}$ (b) $t^{3/4}$ (c) $t^{3/2}$ (d) t^2
6. **A pump can take out 36000 kg of water per hour from a 100 m deep well. It has efficiency of 50%, its power is (g = 10 m/s²).**

   (a) 5 kW (b) 10 kW (c) 15 kW (d) 20 kW

[Answers : (1) 1.33 watt (2) (a) n^2 (b) n^3 (3) a (4) c (5) c (6) d]

C9A Curvelinear Motion

The velocity of a particle traversing curved path, can change both in mgnitude as well as in direction. Remember that the velocity vectos is always tangential to the path. The force which is responsible to change the magnitude of velocity vector i.e. speed is known as tangential force (F_t) where the force which is responsible to change the direction of velocity vector is known as centripetal force (F_c).Centripetal force is also known as normal force or radial force. Tangential force acts along the tangent whereas the centripetal force directed towards the centre as shown in figure.



According to Newton's second law,

the net force along the tangent $\sum \mathbf{F_t} = \mathbf{ma_t} = \mathbf{m\frac{dv}{dt}}$

the net force along the normal $\sum \mathbf{F_c} = \mathbf{ma_c} = \mathbf{\frac{mv^2}{R}}$

Here a_t : tangential acceleration

a_c : centripetal acceleration

V : speed of the particle

R : radius of curvature of the path

Hence the net force acting on the particle

$\mathbf{F = \sqrt{F_t^2 + F_c^2} = m\sqrt{a_t^2 + a_c^2} = ma}$ where $\mathbf{a = \sqrt{a_t^2 + a_c^2}}$ is the net acceleration. Circular motion and projectile motion are the examples of curvelinear motion.

C9B Uniform Circular Motion

When a particle moves in a circle with constant speed, the motion is called uniform circular motion. There is no tangential acceleration i.e.,

$$\mathbf{a_t = \frac{dv}{dt} = 0 \text{ but } a_c = \frac{v^2}{R}}$$

In uniform circular motion the acceleration is perpendicular to the velocity at each instant; as the direction of the velocity changes, the direction of this acceleration also changes. The centripetal acceleration (a_c) at each point in the circular path is directed toward the centre of the circle. We can also express the magnitude of the acceleration in uniform circular motion in terms of the period T of the motion, the time for one revolution

$$\mathbf{T = \frac{2\pi R}{v}} \Rightarrow a_c = \mathbf{\frac{4\pi^2 R}{T^2}}$$

If the speed varies, we call the motion is non-uniform circular motion. In this case

$$\mathbf{a_t = \frac{dv}{dt} \neq 0 \text{ and } a_c = \frac{v^2}{R}}$$

Cetrifugal force is a pseudo force, which is experienced by a non-inertial observer moving in a circular path with constant speed. Its magnitude is equal to that of the centripetal force but its direction is exactly opposite to that.

<u>**Practice Problems :**</u>

1. **Find the angle through which a cyclist bends when he covers a circular path 34.3 m long in √22 sec. Given g = 9.8 ms^{-2}.**

2. **A body of mass 0.5 kg is whirled in a vertical circle by a string 1 m long. Calculate (i) minimum speed it must have at the bottom of the circle so that the string may not slack when the body reaches the top. (ii) In that case, will the velocity at the top of the circle be zero ?**

3. **A stone of mass 0.3 kg tied to the end of a string in a horizontal plane is whirled around in a circle of radius 1 m with a speed of 40 rev./min. What is the tension in the string ? What is the maximum speed with which the stone can be whirled around if the string can withstand the maximum tension of 200 N ?**

4. **A boy is sitting on the horizontal platform of a joy wheel at a distance of 5 m from the centre. The joy wheel begins to rotate and when the angular speed exceeds 10 r.p.m., the boy just slips. What is the coefficient of friction between the boy and the platform ?**

5. **A bucket containing water is tied to one end of a rope 2.45 m long and rotated about the other end in a vertical circle. Find the minimum velocity at the highest and lowest points in order that water in the bucket may not spil.**

6. **A motor cyclist loops the loop whose diameter is 8 m. From what minimum height must be start in order to roll down and go around the loop ? If the cyclist has the mass of 60 kg then find the speed, force exerted by the loop, and net force acting on the cyclist at the following points :**

   (i) at the lowermost point

   (ii) at the highest points

7. **A simple pendulum of length l has a bob of mass m. It is displaced through an angle of θ from the vertical and then released. Choose the incorrect option**

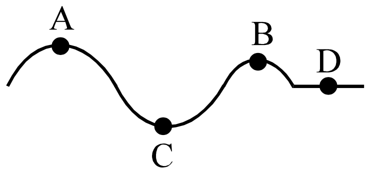
   (a) The speed of the bob at the lowest most point is $\sqrt{2gl(1-\cos\theta)}$

   (b) Tension in the string at the lowest most point is $(3 - 2\cos\theta)mg$

   (c) Tension in the string at the point where it is released is 0

   (d) The centripetal force at the point where it is released is zero.

8. **A car moves at a constant speed on a road as shown in figure. The normal force by the road on the car is N_A, N_B, N_C and N_D when it is at the points A, B, C and D respectively.**



   (a) $N_A = N_B = N_C = N_D$ (b) $N_A > N_B > N_C > N_D$

   (c) $N_A < N_B < N_C < N_D$ (d) $N_C > N_D > N_A > N_B$

9. **A simple pendulum has a string of length l and bob of mass m. When the bob is at its lowest position, it is given the minimum horizontal speed necessary for it to move in a circular path about the point of suspension. The tension in the string at the lowest position of the bob is**

   (a) 3 mg (b) 4 mg (c) 5 mg (d) 6 mg

10. **The speed at which a car can run round a curve of 30 m radius on a level road if the coefficient of friction between the tyres and the road is 0.4, is**

    (a) 5 m/s (b) 10 m/s (c) 20 m/s (d) both (a) and (b)

11. **Keeping the banking angle same, to increase the maximum speed with which a vehicle can travel on a circular road by 10%, the radius of curvature of the road has to be changed from 20 m to**

    (a) 16 m (b) 18 m (c) 24.2 m (d) 30.5 m

12. **A particle is moving along a circular path with a power, directly proportional to time. The tangential force acting on the particle is directly proportional to t^n. Then the value of n is**

    (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{3}{2}$

13. **A particle is projected with velocity u at an angle θ with the horizontal in a gravitational field of intensity E (this is also called the acceleration due to gravity). The field intensity is vertical and acting downward. The time after which the tangential acceleration of the particle becomes zero is**

    (a) $\frac{2u\sin\theta}{E}$ (b) $\frac{3u\sin\theta}{2E}$ (c) $\frac{u\sin\theta}{3E}$ (d) $\frac{u\sin\theta}{E}$

14. **A particle is projected with speed u at an angle θ with the horizontal in vertical plane. The radius of curvature of the path traversed by the particle at the highest point of its trajectory is**

    (a) $\frac{u^2\cos^2\theta}{g}$ (b) $\frac{u^2\cos^2\theta}{2g}$ (c) $\frac{u^2\cos^2\theta}{3g}$ (d) $\frac{u^2\cos^2\theta}{4g}$

[Answers : (1) 45^0 (2) (i) 7 ms^{-1} (ii) 3.1 ms^{-1} (3) 200 N, 25.8 ms^{-1} (4) 0.56 (5) 10.78 ms^{-1} (6) 10 m (7) c (8) d (9) d (10) d (11) c (12) a (13) d (14) a]